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Vector Fundamentals

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PDH308

2 Hours

Course Material and Final Exam

Introduction

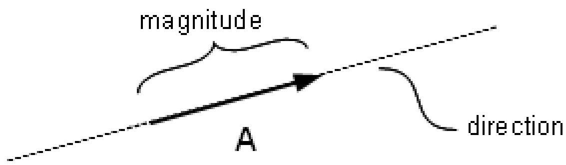
Mechanics is the science of motion and the study of the action of forces on bodies. Mechanics is a physical science incorporating mathematical concepts directly applicable to many fields of engineering such as mechanical, civil, structural and electrical engineering.

Vector analysis is a mathematical tool used in mechanics to explain and predict physical phenomena. The word “vector” comes from the Latin word *vectus* (or *vehere* – meaning to carry). A vector is a depiction or symbol showing movement or a force carried from point A to point B.

A scalar is a quantity, like mass (14 kg), temperature (25°C), or electric field intensity (40 N/C) that only has magnitude and no direction. On the other hand, a vector has both magnitude and direction. Physical quantities that have magnitude and direction can be represented by the length and direction of an arrow. The typical notation for a vector is as follows:

$$\vec{\mathbf{A}} \text{ or simply } \mathbf{A}$$

Note: vectors in this course will be denoted as a boldface letter: **A**.

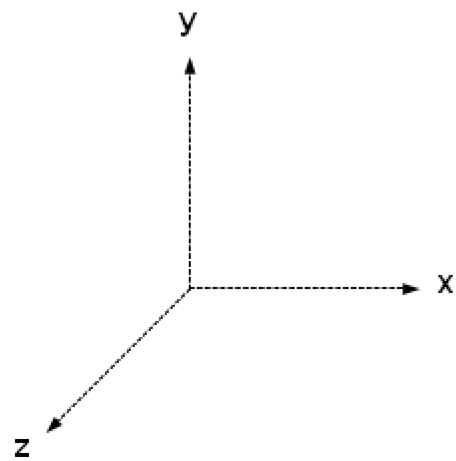


Vectors play an important role in physics (specifically in kinematics) when discussing velocity and acceleration. A velocity vector contains a scalar (speed) and a given direction. Acceleration, also a vector, is the rate of change of velocity.

Vector Decomposition

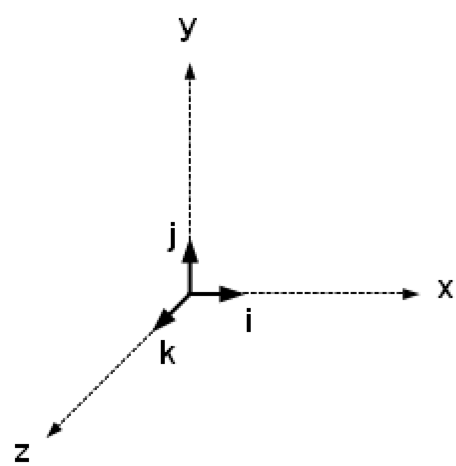
Cartesian Coordinate System

Consider a 3-dimensional Cartesian coordinate system:

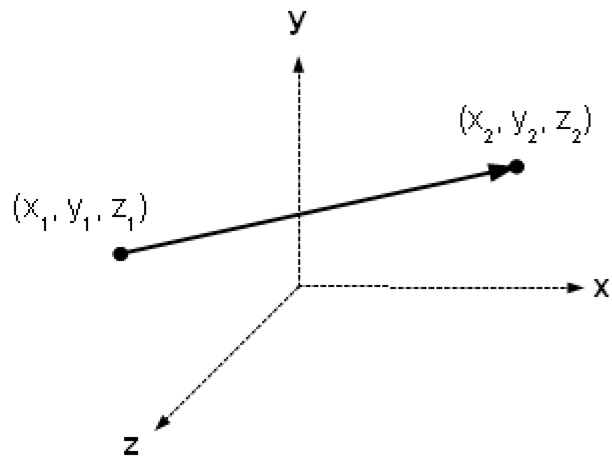


A Cartesian (or rectangular) coordinate system has three mutually perpendicular axes: x, y and z. A vector in this coordinate system will have components along each axes.

A unit vector is a vector along an axes (x, y or z) with a length of one. Let the unit vector along the x-axis be **i** and the unit vector along the y-axis be **j** and the unit vector along the z-axis be **k**. The Cartesian coordinate system with three unit vectors is shown in Figure 3.



A vector can connect two points in space as in Figure 4.

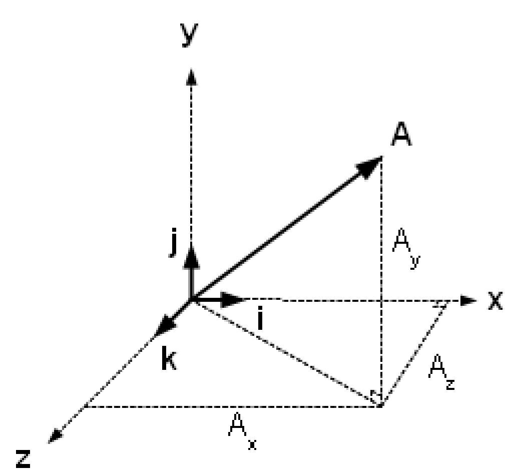


Components of a Vector

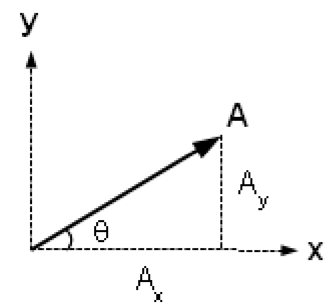
In a Cartesian coordinate system the components of a vector are the projections of the vector along the x, y and z axes. Consider the vector **A**. The vector **A** can be broken down into its components along each axis: A_x, A_y and A_z in the following manner:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

Note that the vectors **i**, **j** and **k** are the unit vectors along each corresponding axis. The unit vectors **i**, **j** and **k** each have a length of one, and the magnitudes along each direction are given by A_x, A_y and A_z.



Trigonometry is utilized to compute the vector components A_x, A_y and A_z. Consider a vector in 2-dimensional space:



The components of this 2-dimensional vector are computed with respect to the angle θ as follows:

$$A_x = A\cos\theta$$

and

$$A_y = A\sin\theta$$

Where A is the magnitude of **A** given by

$$A = \sqrt{A_x^2 + A_y^2}$$

For example, let

$$A = 5 \text{ and } \theta = 36.8°$$

then

$$A_x = 5\cos(36.8°)$$
$$= 5(0.8)$$
$$= 4$$

and

$$A_y = 5\sin(36.8°)$$
$$= 5(0.6)$$
$$= 3$$

Therefore, the vector (in rectangular form) is

$$\mathbf{A} = 4\mathbf{i} + 3\mathbf{j}$$

As a result of the Pythagorean Theorem from trigonometry the magnitude of a vector may be calculated by

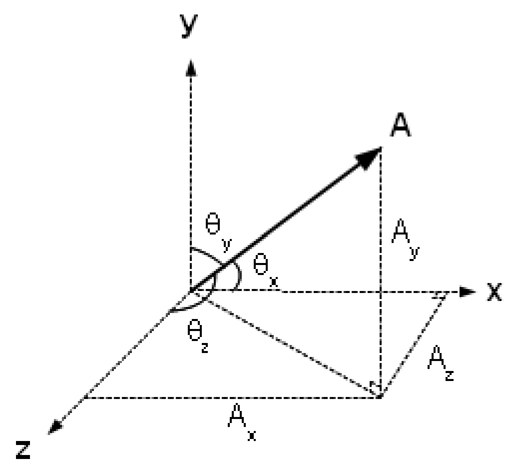
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

The magnitude may also be denoted as

$$A = |\mathbf{A}| \quad \text{or} \quad A = \|\mathbf{A}\|$$

The magnitude of a vector is the length of the vector. It is a scalar (length only) with no direction. In physics, for example, speed is a scalar and velocity is a vector, so speed is the magnitude of the velocity vector.

Now consider, once again, the vector in 3-dimensional space:



The components of this 3-dimensional vector are computed with respect to the angles θ_x, θ_y and θ_z as follows:

$$A_x = A\cos\theta_x$$

$$A_y = A\cos\theta_y$$

$$A_z = A\cos\theta_z$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ is the magnitude of **A** and

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1.$$

A unit vector can be constructed along a vector by dividing the vector by its magnitude. The result is a vector along the same direction as the original vector with magnitude 1. Consider the unit vector **a**:

$$\mathbf{a} = \frac{\mathbf{A}}{A}$$

$$\mathbf{a} = \frac{A_x\mathbf{i}}{A} + \frac{A_y\mathbf{j}}{A} + \frac{A_z\mathbf{k}}{A}$$

Properties of a Vector

Addition

Vector addition is accomplished by adding the components (A_x, A_y and A_z) of one vector to the associated components (B_x, B_y and B_z) of another vector:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

For example, consider the following vectors **A** and **B**:

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$

$$\mathbf{B} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

then

$$\mathbf{A} + \mathbf{B} = (1 + 3)\mathbf{i} + (2 + 1)\mathbf{j} + (5 + 2)\mathbf{k}$$

$$= 4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$$

Commutative Property

Vector addition follows the commutative property:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Associative Property

Vector addition also follows the associative property:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

Scalar Multiplication

Vectors can be multiplied by real numbers (called scalars). To accomplish this the vector components (A_x, A_y and A_z) are each multiplied by the real number (n):

$$n\mathbf{A} = nA_x\mathbf{i} + nA_y\mathbf{j} + nA_z\mathbf{k}$$

For example, let n = 5, and

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$

then

$$n\mathbf{A} = 5\mathbf{i} + 10\mathbf{j} + 25\mathbf{k}$$

Commutative Property

Scalar multiplication of a vector follows the commutative property:

$$n\mathbf{A} = \mathbf{A}n$$

(where n is a scalar)

Associative Property

Scalar multiplication of a vector follows the associative property:

$$(ab)\mathbf{A} = a(b\mathbf{A})$$
$$= (ba)\mathbf{A}$$
$$= b(a\mathbf{A})$$

(where a and b are scalars)

Distributive Property

Scalar multiplication of a vector follows the distributive property:

$$(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$$
$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$$

(where a and b are scalars)

Identity Vector and Zero Vector

$$1\mathbf{A} = \mathbf{A}$$
$$0 + \mathbf{A} = \mathbf{A}$$

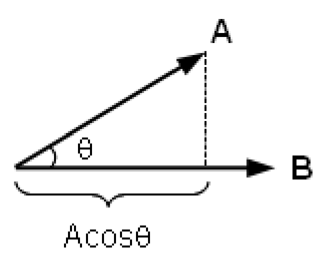
Dot Product

The dot product (or scalar product, or inner product) is a vector operation that takes two vectors and generates a scalar quantity (a single number). The dot product is used to obtain the cosine of the angle between two vectors.

The dot product is denoted by the following notation:

$$\mathbf{A} \bullet \mathbf{B}$$

Figure 8 shows two vectors in space with an angle θ between the two vectors:



The dot product is defined by

$$\mathbf{A} \bullet \mathbf{B} = AB\cos\theta$$

where

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

The dot product between two vectors **A** and **B** is also defined as

$$\mathbf{A} \bullet \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

or generally as

$$\mathbf{A} \bullet \mathbf{B} = \Sigma A_iB_i$$

(where i is defined from 1 to n, where n is the number of dimensions)

The above definition is derived from the following expansion, since the base unit vectors are orthogonal:

$$\begin{aligned}\mathbf{A} \bullet \mathbf{B} &= (A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}) \bullet (B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}) \\ &= A_xB_x\,\mathbf{i} \bullet \mathbf{i} + A_xB_y\,\mathbf{i} \bullet \mathbf{j} + A_xB_z\,\mathbf{i} \bullet \mathbf{k} + \\ &\quad A_yB_x\,\mathbf{j} \bullet \mathbf{i} + A_yB_y\,\mathbf{j} \bullet \mathbf{j} + A_yB_z\,\mathbf{j} \bullet \mathbf{k} + \\ &\quad A_zB_x\,\mathbf{k} \bullet \mathbf{i} + A_zB_y\,\mathbf{k} \bullet \mathbf{j} + A_zB_z\,\mathbf{k} \bullet \mathbf{k}\end{aligned}$$

Since the base unit vectors **i**, **j** and **k** are orthogonal (or perpendicular):

$$\mathbf{i} \bullet \mathbf{i} = 1 \qquad \mathbf{i} \bullet \mathbf{j} = 0$$

$$\mathbf{j} \bullet \mathbf{j} = 1 \qquad \mathbf{i} \bullet \mathbf{k} = 0$$

$$\mathbf{k} \bullet \mathbf{k} = 1 \qquad \mathbf{j} \bullet \mathbf{k} = 0$$

then the dot product reduces to

$$\mathbf{A} \bullet \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

For example, let

$$\mathbf{A} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$$

then the dot product of the two vectors is

$$\begin{aligned}\mathbf{A} \bullet \mathbf{B} &= 6 + 15 - 4 \\ &= 17\end{aligned}$$

Applications of the Dot Product

Angle formed Between Two Vectors

To find the angle formed by two vectors use the definition of the dot product:

$$\mathbf{A} \bullet \mathbf{B} = AB\cos\theta$$

and

$$\mathbf{A} \bullet \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

Setting these two equalities equal to each other provides the following result:

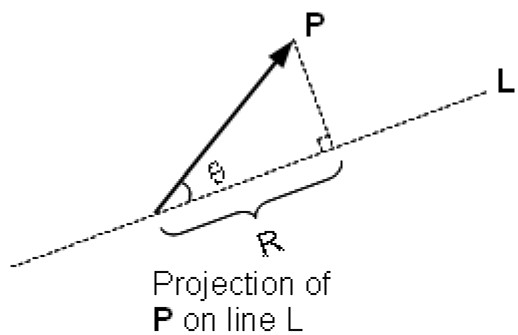
$$AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

or

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$$

Projection of a vector onto a Line

Consider a vector **P** forming an angle θ with a line. The projection of **P** on the line is also called the orthogonal projection.

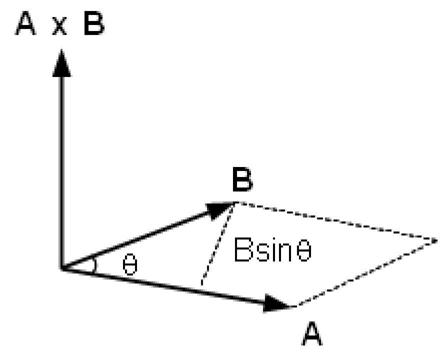


The projection of **P** on line L is given by

$$R = P\cos\theta$$

Cross Product

The cross product (or vector product, or outer product) of two vectors results in a vector perpendicular to both vectors. The magnitude of the resulting vector is equal to the area of the parallelogram generated by the two vectors. The area of a parallelogram equals the height times the base, which is a magnitude of the cross product.



The cross product is denoted by

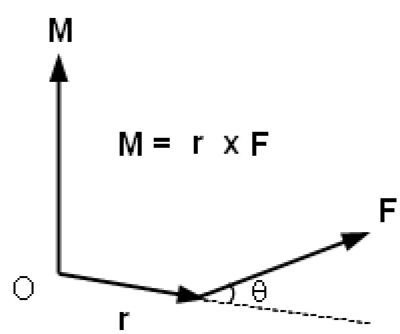
$$\mathbf{A} \times \mathbf{B}$$

The cross product is defined by

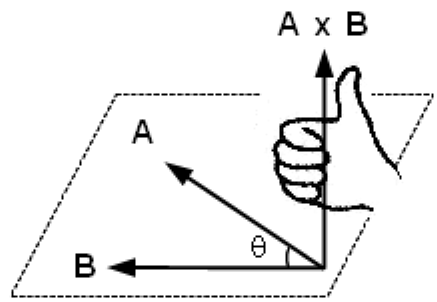
$$\mathbf{A} \times \mathbf{B} = AB\sin\theta\ \mathbf{n}$$

(where θ is the angle between the vectors and **n** is the unit vector normal or perpendicular to **A** and **B**)

The name cross product is derived from the cross symbol "×" that is used to designate its operation. The name vector product emphasizes the vector nature of the result, instead of a scalar. The cross product has many applications in mathematics, physics and engineering such as the moment of a force about a point (or torque).



The cross product obeys the right-hand rule as in Figure 12.



The right-hand rule is used to determine the direction of the resulting vector. The vectors A and **B** form a plane. The vector formed from the resulting cross product **A** × **B** is in a direction perpendicular to the plane of the two vectors. Using the right-hand rule will determine if the direction of the vector is above the plane of the vectors or below the plane. Using your right hand, place your hand above the plane of the vectors at their vertex. Curl your fingers in the direction from **A** to **B**. If necessary, turn your hand over so that your thumb points down through the plane in order to curl your fingers from **A** to **B**. The resulting vector points in the direction of your thumb, either up or down.

Computation of the Cross Product

The cross product may be computed by multiplying the components of the vectors or by assembling the components along with their unit vectors into a matrix and taking the determinant of the matrix.

Cross Product by Multiplying Components

$$\mathbf{A} \times \mathbf{B} = (A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}) \times (B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k})$$

Consider the cross product of the unit vectors **i**, **j** and **k**

$$\mathbf{i} \times \mathbf{i} = 0 \qquad \mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{j} = 0 \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{k} = 0 \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$
$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

Therefore,

$$\mathbf{A} \times \mathbf{B} = (A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}) \times (B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k})$$
$$= A_xB_x\,\mathbf{i}\times\mathbf{i} + A_xB_y\,\mathbf{i}\times\mathbf{j} + A_xB_z\,\mathbf{i}\times\mathbf{k} +$$
$$A_yB_x\,\mathbf{j}\times\mathbf{i} + A_yB_y\,\mathbf{j}\times\mathbf{j} + A_yB_z\,\mathbf{j}\times\mathbf{k} +$$
$$A_zB_x\,\mathbf{k}\times\mathbf{i} + A_zB_y\,\mathbf{k}\times\mathbf{j} + A_zB_z\,\mathbf{k}\times\mathbf{k}$$

$$= A_xB_x\,(\mathbf{0}) + A_xB_y\,\mathbf{k} + A_xB_z\,(-\mathbf{j}) +$$
$$A_yB_x\,(-\mathbf{k}) + A_yB_y\,(\mathbf{0}) + A_yB_z\,\mathbf{i} +$$
$$A_zB_x\,\mathbf{j} + A_zB_y\,(-\mathbf{i}) + A_zB_z\,(\mathbf{0})$$
$$= (A_yB_z - A_zB_y)\,\mathbf{i} + (A_zB_x - A_xB_z)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k}$$

or, graphically, and perhaps easier to remember:

$$\overrightarrow{\mathbf{i}\ \ \mathbf{j}\ \ \mathbf{k}\ \ \mathbf{i}\ \ \mathbf{j}}$$

Cross Product by Matrix Method

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\,\mathbf{i} + (A_zB_x - A_xB_z)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k}$$

For example, let

$$\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

and

$$\mathbf{B} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Then the cross product of the two vectors is

$$= -12\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} - (3\mathbf{k} + 10\mathbf{i} + 8\mathbf{j})$$
$$= -22\mathbf{i} - 13\mathbf{j} + \mathbf{k}$$

Properties of the cross product

Anti-Commutative

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

Scalar Multiplication

$$\mathbf{nA} \times \mathbf{B} = \mathbf{n(A} \times \mathbf{B)}$$

Vector Addition and the Cross Product

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}$$

and

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

Triple Product

The triple product is the dot product of a vector with the result of the cross product of two other vectors.

$$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$$

The triple product may be computed by taking the determinant of the following matrix:

$$\mathbf{A} \bullet \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \bullet (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B})$$

Summary

Vector mechanics is the application of vectors in the science of mechanics. Mechanics is the science of motion and the study of the action of forces on bodies. Vector analysis is very important in many fields of engineering such as mechanical, civil, structural and electrical engineering.

Scalar values, such as mass and temperature convey only a magnitude, but vectors such as velocity employ both a magnitude and a direction. The dot product is a vector operation on two vectors that produces a scalar value. The dot product is used to find the angle between two vectors or to find the projection of a vector onto a line. The cross product is a vector operation on two vectors that produces another vector. The cross product may be used to calculate the moment of a force around a point at a given radius.

References

1. Beer, Ferdinand P. and Johnston, E Russell Jr. *Vector Mechanics for Engineers: Statics, Fifth Edition*. New York, New York: McGraw-Hill Book Company, 1988.

2. “Cross Product”. 24 January 2012 <http://en.wikipedia.org/wiki/Cross_product>

3. Davis, Harry F. and Snider Arthur David. *Introduction to Vector Analysis, Seventh Edition*. Dubuque, IA: Wm C. Brown Publishers: 1995.

4. “Dot Product”. 30 December 2011 <http://en.wikipedia.org/wiki/Dot_product>

5. “Euclidean Vector”. 26 January 2012 <http://en.wikipedia.org/wiki/Euclidean_vector>

6. “Review A: Vector Analysis”. visited 27 December 2011 <http://web.mit.edu/8.02t/www/materials/modules/ReviewA.pdf>

7. “Vector Methods”. visited 27 December 2011 <http://emweb.unl.edu/math/mathweb/vectors/vectors.html>

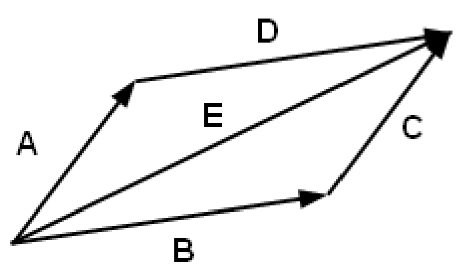
Final Exam – Vector Fundamentals

1. _____________ are properties of a vector.
    a. Magnitude and direction
    b. Weight and mass
    c. Speed and size
    d. Latitude and longitude

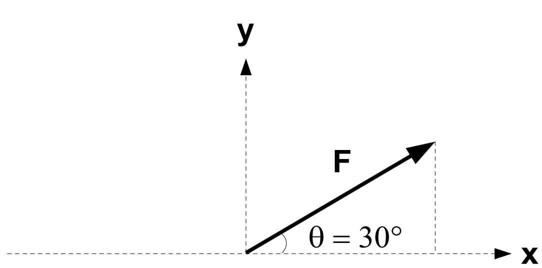
2. A scalar has both magnitude and direction.
    a. True
    b. False

3. All of the following are examples of vectors except _____________.
    a. temperature
    b. velocity
    c. force
    d. moment (torque)

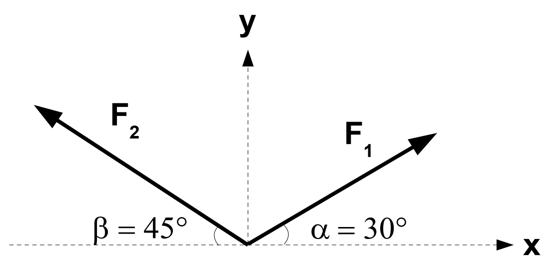
4. Given vector **A** = 3**i** + 4**j** – 5**k**, its length is _____________.
    a. 7.07
    b. 12
    c. 3.46
    d. 5.02



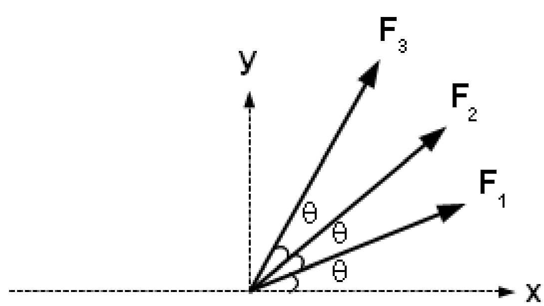
5. Using the above diagram, A + B equals _____________.
    a. C
    b. E
    c. D
    d. none of the above



6. Using the above diagram, if F = 70 N, the vector component along the x-axis equals ___________N and the vector component along the y-axis equals ___________N.
    a. 41.3, 56.5
    b. 35, 60.6
    c. 50.1, 48.9
    d. 60.6, 35



7. Using the above diagram, if F1 = 60 N, F2 = 50 N, the resulting force vector is F = _____________ i + _____________ j N.
    a. 13.4, 63.2
    b. 87.4, 65.4
    c. 16.6, 65.3
    d. 18.3, 62.1

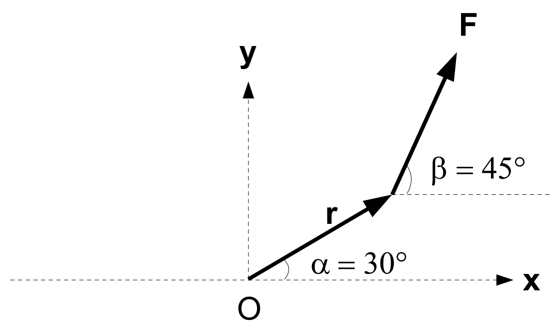


8. Using the above diagram, if the angles all equal 20 degrees, F1 = 10 N, F2 = 12 N, F3 = 15 N, the resultant vector is F = _____________ i + _____________ j N.
    a. 26.1, 24.1
    b. 7.50, 13.0
    c. 27.1, 22.6
    d. 9.40, 3.42

9. Given that **A** = 4**i** + 2**j** – 5**k** and **B** = **i** – 7**j** – 5**k**, the dot product **A • B** is _____________.
    a. 14
    b. 15
    c. 16
    d. 17

10. Given that **A** = 4**i** + 2**j** – 5**k** and **B** = **i** – 7**j** – 5**k**, the cross product **A** × **B** is ____________.
    a. –43**i** + 17**j** + 18**k**
    b. 4**i** – 14**j** + 25**k**
    c. 12**i** – 23**j** + 27**k**
    d. –45**i** + 15**j** – 30**k**

11. The angle between **A** and **B** in the above problem is ____________.
    a. 74.1
    b. 78.6
    c. 75.0
    d. 68.3



12. Using the above diagram, the moment **M** resulting from the force **F** = 10 N applied at a distance **r** = 5m from the point O is ____________ **k** Nm.
    a. 11.2
    b. 12.9
    c. 13.5
    d. 14.8

13. Two vectors, **A** and **B**, form an angle of θ. The projection of **A** on **B** is given by ____________.
    a. Bcosθ
    b. **A • B**
    c. **A** × **B**
    d. **(A • B)** / B

14. The angle between two known vectors may be found using ____________.
    a. parallelogram method
    b. vector decomposition
    c. the triple product
    d. the dot product

15. The right-hand rule is used ____________.
    a. to calculate the dot product of two vectors
    b. to find the projection of a vector onto another vector
    c. to determine the direction of the resulting vector from a cross product

d. to find the magnitude of the resulting vector from a cross product