Electrical Power

Part IV: Transmission Lines

by

John A Camara, BS, MS, PE, TF

Course 533

4 PDH (4 Hours)

PO Box 449
Pewaukee, WI 53072
(888) 564 - 9098
eng-support@edcet.com

Nomenclature[1]

A	ABCD parameter	-
a	phase	-
A	area	m^2
B	ABCD parameter	-
B	magnetic flux density	T
B	magnetic flux density	T
B	susceptance	S, Ω^{-1}, or mho
b	phase	-
c	speed of light	m/s
C	capacitance	F
C	ABCD parameter	-
c	phase	-
D	ABCD parameter	-
D	distance	m
E	electric field strength	V/m
E	energy	J
E	voltage (generated)	V
f	frequency	Hz, s^{-1}, cycles/s
f_{droop}	frequency droop	Hz/kW
G	conductance	S, Ω^{-1}, or mho
GMD	geometric mean distance	m
GMR	geometric mean radius	m
h	specific enthalpy	kJ/kg
I	effective or DC current	A
I	rms phasor current	A
K	correction factor	-
K	skin effect ratio	-
l	length	m
L	inductance	H
m	mass	kg

[1] Not all the nomenclature, symbols, or subscripts may be used in this course—but they are related, and may be found when reviewing the references listed for further information. Further, all the nomenclature, symbols, or subscripts will be found in all "Parts" of this complete course. For guidance on nomenclature, symbols, and electrical graphics: IEEE 280-2021. IEEE Standard Letter Symbols for Quantities Used in Electrical Science and Electrical Engineering. New York: IEEE; and IEEE 315-1975. Graphic Symbols for Electrical and Electronics Diagrams. New York: IEEE, approved 1975, reaffirmed 1993.

M	mutual inductance	H
n	Steinmetz exponent	-
N	number of turns	-
n_s	synchronous speed	r/min or min^{-1}
p	pressure	Pa
P	number of poles	-
P	power	W
pf	power factor	-
pu	per unit	-
Q	heat	J
r	radius	m
R	resistance	Ω
s	specific entropy	$\text{kJ/kg}\cdot\text{K}$
S	apparent power	kVA
SWR	standing wave ratio	-
T	temperature	ºC or K
v	wind velocity	km/hr
V	effective or DC voltage	V
v	velocity (speed)	m/s
V	rms phasor voltage	V
V_{droop}	voltage droop	V/kVAR
VR	voltage regulation	-
W	work	kJ
X	reactance	Ω
x	variable	-
Y	admittance	S, Ω^{-1}, or mho
y	admittance per unit length	S/m, $1/\Omega\cdot\text{m}$ Ω^{-1}, or mho/m $[\mho/\text{m}]$
Z	impedance	Ω
z	impedance per unit length	Ω/m
Z_0	characteristic impedance	Ω

Symbols

α	turns ratio	-
α	attenuation constant	Np/m
α	thermal coefficient of resistance	1/°C
β	phase constant	rad/m
γ	propagation constant	rad/m
Γ	reflection coefficient	-
δ	skin depth	m
Δ	change, final minus initial	-
ε	permittivity	F/m
ε_0	free-space permittivity	8.854×10^{-12} F/m
ε_r	relative permittivity	-
η	efficiency	-
θ	phase angle	rad
κ	coupling coefficient	-
μ	permeability	H/m
μ_o	free-space permeability	1.2566×10^{-6} H/m
μ_r	relative permeability	-
ξ	ratio of radii	-
ρ	resistivity	$\Omega \cdot m$
σ	conductivity	S/m
ω	armature angular speed	rad/s

Subscripts

ϕ	phase
0	zero sequence
0	characteristic
0	free space (vacuum)
0,o	initial (zero value)
1	positive sequence
1	primary
2	negative sequence
2	secondary
ab	a to b
AC	alternating current
avg	average
bc	b to c
c	controls or critical
c	core
C	capacitor
ca	c to a
Cu	copper
d	direct
DC	direct current
e	eddy current
e	equivalent
eff	effective
ext	external
f	final / frequency
fl	full load
g	generator
h	hysteresis
int	internal
l	line
l	per unit length
L	inductor

ll	line-to-line
m	motor
m	maximum
m	mutual
max	maximum
n	neutral
nl	no load
O	origin
oc	open circuit
p	phase
p	primary
ps	primary to secondary
pu	per unit
q	quadrature
R	receiving end
R	resistance
s	synchronous
s	secondary
S	sending end
sc	short circuit
sys	system
t	terminal
w	wave

TABLE OF CONTENTS

List of Figures

List of Tables

List of Examples

List of Equations

INTRODUCTION

Although this is a five part course, each individual part is meant to be stand-alone should one be interested in that topic. The overall purpose of the course is to provide an overview of electric power from generation, through the various distribution systems, including the vital transformer links that change the voltage from the high voltage required for minimum losses during transmission to medium- and low-voltage for the end-users. Additionally, the transmission lines connecting the system are covered. And, finally, the rule from the National Electric Safety Code® (NESC®) that govern it all completes the overview.

Part I, Generation, the more common type of plants producing the power. The basics of alternating current and direct current generators is explained include the principles of parallel operation. Finally, energy management and power quality are covered.

Part II, Distribution Systems, covers the classification of such systems, how the common neutral is utilized, overhead and underground distribution, along with fault analysis methods.

Part III, Transformers, informs on power transformers, their ratings, voltage regulation, testing methods and parameters used to analyze both transformers and transmission lines.

Part IV, Transmission Lines, discusses the electrical parameters of such line: resistance, inductance, and capacitance. Important effects such as the skin effect and reflection are explained. This part completes with an explanation of models for each type of transmission line: short, medium, and long.

Part V, The National Electrical Safety Code, covers organization of the code and some of the multitude of requirements for the transmission of electrical power.

The information is primarily from the author's books, Refs. [A] and [B] with the NESC information from the Handbook covering the code, Ref. [C]. The coverage of the NESC does not include end-users buildings—this is covered by the NEC, Ref. [D]. Information useful in many aspects of electric engineering may be found in [E] and [F] as well as the appendices. Reference [G] has detailed descriptions of analysis techniques. Reference [H] provides detailed engineering review for Parts I through IV of this course. Reference [I] provides indepth explanation of the per-unit system often used in such engineering.

FUNDAMENTALS

A *power transmission line* is a system of conductors that is designed to transfer power from a source to a load. Conductors used in power transmission lines include wires, coaxial cables and lines, and waveguides. A *coaxial cable* or *coaxial line* is one in which the conductor is centered inside, and insulated from, an outer metal tube that acts as a second conductor. Coaxial cables and lines are used at frequencies up to approximately 1 GHz; above 1 GHz, losses become prohibitive and waveguides are used instead. *Waveguide*s are devices, such as hollow metal structures of various shapes, that confine electromagnetic waves to particular paths as the waves travel. Waveguides are used at frequencies up to approximately 100 GHz; above 100 GHz, losses become prohibitive and optical guides are used that can operate at frequencies of approximately 2×10^{14} Hz with minimal losses.

The wavelength that is transmitted by a power transmission line must be many times larger than the length of the line itself, or else the power will be radiated into space. For example, a power transmission line operating at 60 Hz will have a wavelength of 5000 km but will rarely have a length greater than 500 km.

Transmission lines that are designed to carry wavelengths that are shorter than or comparable to the length of the line are called *high-frequency transmission lines*. These lines have different properties from power transmission lines and are often used in communication systems for the transmission of signals. Waveguides are used in high-frequency transmission for wavelengths less than approximately 10 cm. A conductor specifically designed with a length comparable to the wavelength is called an *antenna*.

A transmission line guides *transverse electromagnetic* (TEM) *waves*. In TEM waves, the electric and magnetic fields are perpendicular to the direction of propagation. At low frequencies, where the dimensions of the line are small compared to the wavelength, lumped parameters can provide an adequate model of the circuit.[2] This is because the displacement current is small and can be ignored. That is, the storage of electromagnetic energy in the space surrounding the line can be ignored. At high frequencies, the dimensions of the line are large compared to the wavelengths

---

[2] Lumped parameters are single-valued circuit elements that are equivalent to the value in the circuit as a whole, such as a resistor, inductor, or capacitor in a specific position in the circuit that accounts for the line electrical properties as well.

propagated. The displacement current becomes significant. That is, electric and magnetic energy is stored in the space surrounding the line.

To avoid using field theory to analyze transmission lines, distributed parameters are used along with currents and voltages associated with the electric and magnetic fields. *Distributed parameters* are circuit elements that exist along the entire line but are represented per unit length, which allows the use of common electrical analytic techniques. A transmission line represented by distributed parameters is illustrated in Fig. 1.

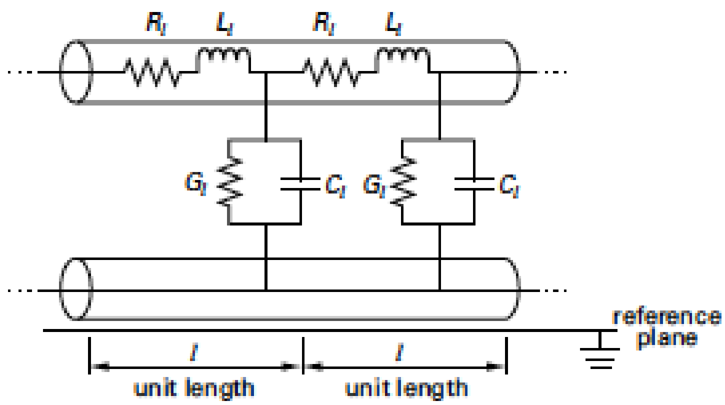


Figure 1: Distributed Parameters

The distributed parameters in Fig. 1 are the resistance per unit length, R_l ,inductance per unit length, L_l, capacitance per unit length, C_l, and shunt conductance per unit length, G_l. The resistance depends on the frequency because of a phenomenon known as the *skin effect*: as frequency rises, current flows closer to the surface of the conductor, which raises the resistance. (The AC resistance at 60 Hz is 5% to 10% higher than the DC resistance.) The *temperature effect* on resistance must also be accounted for in transmission line design. The inductance consists of two terms, the internal inductance, L_{int}, and the external inductance, L_{ext}. The internal inductance depends on the skin effect, decreasing as the frequency increases. The external inductance depends on the geometric arrangement of the conductors. The capacitance also depends on the geometric arrangement of the conductors. The *shunt conductance*, also called the *line-to-line conductance*, is usually insignificant.

Additional parameters of concern include the characteristic impedance, the standing wave ratio, and the reflection coefficient. The *characteristic impedance* is determined by Eq. 1.

Equation 1: Characteristic Impedance

$$Z_0 = \sqrt{\frac{Z_l}{Y_l}} \quad \text{[In W]}$$

If the transmission line is terminated with the characteristic impedance, no power is reflected back to the source. If the termination impedance is other than Z_0, then the power (signal) from the generator (source) will be partially reflected back to the generator. In wave terminology, the generator's waves will then combine in the transmission line with the reflected waves to form *standing waves*. The *standing wave ratio,* SWR, is the ratio of the maximum to the minimum voltages (currents) encountered along the transmission line. Typically, the SWR is greater than unity. If the terminating impedance matches the characteristic impedance, all the input power provided by the generator will be absorbed by the load and the SWR will equal one. The standing wave ratios are as follows.

Equation 2: Standing Wave Ratios

$$\text{SWR} = \text{VSWR} = \text{CSWR} = \max \begin{cases} Z_{\text{load}} / Z_0 \\ Z_0 / Z_{\text{load}} \end{cases}$$

The *reflection coefficient*, Γ, is the ratio of the reflected voltage (current) to the incident voltage (current). The reflection coefficient is Given in Eq. 3.

Equation 3: Reflection Coefficient

$$G = \frac{V_{\text{reflected}}}{V_{\text{incident}}} = \frac{I_{\text{reflected}}}{I_{\text{incident}}} = \left| \frac{Z_{\text{load}} - Z_0}{Z_{\text{load}} + Z_0} \right|$$

The fraction of incident power that is reflected back to the source from the load is Γ^2. The relationship between the reflection coefficient and the standing wave ratio is given by Eq. 4.

Equation 4: Reflection Coefficient--Standing Wave Ratio

$$G = \frac{\text{SWR} - 1}{\text{SWR} + 1}$$

The *velocity of propagation*, v_w, is the velocity at which a wave propagates along a transmission line, given by Eq. 5.

Equation 5: Propagation Velocity

$$v_w = \frac{1}{\sqrt{L_l C_l}}$$

Transmission lines are considered in three lengths.[3] The *short transmission line* is from 0 km to 80 km (0 mi to 50 mi). In a short line, the shunt parameters, G_l and C_l, are insignificant and ignored in the analysis. The *medium transmission line* is from 80 km to 240 km (50 mi to 150 mi). In a medium line, the shunt capacitances are generally lumped in predetermined locations along the line. The *long transmission line* is greater than 240 km (150 mi). A long transmission line is also called a *uniform transmission line* because it has constant distributed parameters along the line from the sending end to the receiving end.

Many transmission line characteristics and formulas are tabulated in English Engineering System units. For this reason, a mixture of metric and English Engineering System units is utilized in this course. Additionally, all AC quantities have a magnitude and an angle. However, as is common practice, these items will be shown in phasor notation, or complex notation, only when possible confusion could occur.[4]

DC RESISTANCE

Resistance generates the I^2R power loss in transmission lines. Additionally, the IR drop, or voltage drop, affects the voltage regulation. The DC resistance, often called the *characteristic resistance*,

[3] The distances given assume a 60 Hz signal. The engineering methods in each type of transmission line differ. As the frequency increases, the distances of each definition decrease, with the concurrent adjustment in the engineering method used.

[4] Phasor or complex notation usually presents the parameter in bold or with an arrow over the top ($\mathbf{V}$ or $\overline{V}$).

R_0, of a conductor of length l, cross-sectional area A, and resistivity ρ is normally given as $\rho l/A$. Because the distributed parameters are given as per-unit-length quantities, the DC resistance per unit length is given as follows.

Equation 6: DC Resistance per unit Length

$$R_{l,\mathrm{DC}} = R_0 = \frac{\rho}{A} \quad [\text{In } \Omega/\text{m}]$$

Resistance sensitivity to temperature is calculated from the following.

Equation 7: Resistance vs. Temperature

$$R_{\mathrm{final}} = R_{\mathrm{initial}}(1 + \alpha \Delta T)$$

Voltage drops are often tabulated for various conductor sizes as shown in Table 1.

Table 1: Voltage Drop in Two-Wire DC Circuits (Loop)*

conductor size (AWG)		
copper	approximately equivalent aluminum	voltage dropper 100,000 A·m
6	4	31.3
4	2	19.7
2	1/0	12.4
1/0	3/0	7.80
2/0	4/0	6.19

*Values are calculated for 32°C (90°F) conductors.

Example 1

A 50×10^3 W DC device is located 1.6 km from a motor-generator with a maximum output voltage of 600 V_{DC}. The lowest voltage the DC device can properly utilize is 575 V.

What is the minimum conductor size required?

Solution

The maximum allowable voltage drop is 25 V. To use Table 1, the total ampere-meters for this system is required. The maximum current is given by

$$\begin{aligned} P &= IV \\ I &= \frac{P}{V} = \frac{50 \times 10^3 \text{ W}}{575 \text{ V}} \\ &= 86.96 \text{ A} \end{aligned}$$

The total cable distance to reach the site *and provide the return path* (thus the times 2) to the generator is

$$D = (2)(1.6 \times 10^3 \text{ m}) = 3.2 \times 10^3 \text{ m}$$

The total ampere-meters is

$$T_{A-m} = (86.96 \text{ A})(3.2 \times 10^3 \text{ m}) = 278.27 \times 10^3 \text{ A} \cdot \text{m}$$

Now, find the voltage drop to use in the table.

$$\begin{aligned} V_{\text{drop,allowable}} &= (A \cdot \text{m})_{\text{total}} \left(\frac{V_{\text{drop,actual}}}{100{,}000 \text{ A} \cdot \text{m}} \right) \\ 25 \text{ V} &= (278.27 \times 10^3 \text{ A} \cdot \text{m}) \left(\frac{V_{\text{drop,actual}}}{100{,}000 \text{ A} \cdot \text{m}} \right) = 2.78 (V_{\text{drop,actual}}) \\ V_{\text{drop,actual}} &= \frac{25 \text{ V}}{2.78} = 8.98 \text{ V} \end{aligned}$$

From Table 1, the conductor size with an 8.98 voltage drop, or less, is the 1/0 for copper and 3/0 for aluminum.

SKIN EFFECT

The *skin effect* is the tendency of AC currents to flow near the surface of a conductor. This results in the AC current being restricted to less than the total cross-sectional area of the conductor, resulting in an increased resistance and a decreased internal inductance. The skin effect can be thought of as an electromagnetic phenomenon related to the time required for an external field to penetrate a conductor.[5] It occurs because the surface reactance is smaller than the reactance of other possible current paths within the conductor. The *skin depth*, δ, is the depth beneath the surface of the conductor that is carrying the current at a given frequency caused by electromagnetic waves incident to the surface.[6] Specifically, it is the depth at which the current density is $1/e$, the value of the surface current density. The skin depth for a flat conducting plate is

Equation 8: Skin Depth for Flat Conducting Plate

$$\delta = \frac{1}{\sqrt{\frac{\pi f \mu}{\rho}}} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

The AC resistance per unit length for a flat conducting plate of unit width w is as follows.

Equation 9: Resistance per unit Length for Flat Conducting Plate

$$R_{l,\text{AC}} = \frac{\rho}{\delta w} \quad [\text{In } \Omega/\text{m}]$$

Conductors are normally round. Equation 9 would be applicable only if the skin depth were much less than the radius, r, so that the wire approximated a flat surface. For example, a ratio of $r/\delta > 5.5$ results in less than a 10% error.

[5] It is this principle that allows metal shielding to be used to prevent electromagnetic wave penetration of equipment. Conversely, such shielding also prevents the escape of radiated electromagnetic waves.
[6] In generators, the armature current is induced from an electromagnetic wave incident to the surface. This principle applies for generators and any conductor cabling attached to the generator.

The actual skin depth for copper conductors is

Equation 10: Skin Depth for Round Copper Conductors

$$d_{\text{Cu}} = \frac{0.066}{\sqrt{f}} \quad \text{[In m]}$$

Example 2

Determine the copper wire radius in mils where the skin depth equals the radius for the following frequencies: 60 Hz, 10 kHz, 1 MHz.

Solution

First convert Eq. 10 into an equivalent equation base on mils rather than meters.

$$d_{\text{Cu}} = \frac{0.066}{\sqrt{f}} \quad \text{[In m]}$$

$$d_{\text{Cu}} = \left(\frac{0.066}{\sqrt{f}}\ \text{m}\right)\left(\frac{1\ \text{mil}}{0.001\ \text{in}}\right)\left(\frac{1\ \text{in}}{2.25\ \text{cm}}\right)\left(100\ \frac{\text{cm}}{\text{m}}\right) = \frac{2.6\times 10^3}{\sqrt{f}} \quad \text{[In mils]}$$

Substitute the given frequencies to determine the skin depth.

$$d_{\text{Cu,60 Hz}} = \frac{2.60\times 10^3}{\sqrt{f}}\ \text{mil} = \frac{2.60\times 10^3}{\sqrt{60\ \text{Hz}}}\ \text{mil} = 336\ \text{mil}$$

$$d_{\text{Cu,10 kHz}} = \frac{2.60\times 10^3}{\sqrt{f}}\ \text{mil} = \frac{2.60\times 10^3}{\sqrt{10\times 10^3\ \text{Hz}}}\ \text{mil} = 26\ \text{mil}$$

$$d_{\text{Cu,1 MHz}} = \frac{2.60\times 10^3}{\sqrt{f}}\ \text{mil} = \frac{2.60\times 10^3}{\sqrt{1\times 10^6\ \text{Hz}}}\ \text{mil} = 2.6\ \text{mil}$$

The largest solid copper wire is AWG 4/0, that is, AWG 0000.[7] The radius is 230 mil. This indicates that at 60 Hz the electromagnetic wave completely penetrates the conductor. Skin effect occurs at 60 Hz but only for the largest conductors, such as those used for commercial power applications (on the order of 500 kcmil). For most other applications, the 10 kHz frequency mark is often used as the point where skin effect must be accounted for and AC resistance calculated.

AC RESISTANCE

Skin effect and the resulting increase in resistance and decrease in internal inductance are often neglected in initial calculations or approximations. In the case of large copper wires at commercial power frequencies, exact calculations must consider the effect.[8] The effect on aluminum conductors is less because aluminum has a higher resistivity. Tables of skin effect ratios are available when the exact value of the AC resistance is required.[9] Table 2 gives sample ratios. The skin effect ratio for the AC resistance, K_R, is used to determine the effective resistance from

Equation 11: Effective AC Resistance

$$R_{\mathrm{AC}} = K_R R_{\mathrm{DC}}$$

[7] Any larger size consists of stranded conductors.
[8] The effect should be accounted for at communications system frequencies as well.
[9] Skin effect ratios are also called AC/DC resistance ratios.

Table 2: Skin Effect Ratios

x	K_R[a]	$K_{L,\text{int}}$[b]
0.0	1.00000	1.00000
0.5	1.00032	0.99984
1.0	1.00519	0.99741
2.0	1.07816	0.96113
4.0	1.67787	0.68632
8.0	3.09445	0.35107
16.0	5.91509	0.17649
32.0	11.56785	0.08835
50.0	17.93032	0.05656
100.0	35.60666	0.02828
∞	∞	0

[a]K_R is the skin effect ratio for AC resistance.
[b]$K_{L,\text{int}}$ is the skin effect ratio for internal inductance.

The table is entered using a value of *x* determine from the following.

Equation 12: Table 2 Entry Argument

$$x = 2pr\sqrt{\frac{2fm}{r}} = \frac{2r\sqrt{2p}}{d}$$

The use of tables is the most accurate practical method of determining the AC resistance. Curves normalized to the DC resistance are sometimes used for intermediate frequencies. The AC resistance can also be found by calculating the effective area of the conductor, that is, the area actually conducting current, and comparing this area to that used by the DC resistance. The DC resistance is given by Eq. 6 as ρ/A. Because the resistivity doesn't change, comparing an AC resistance to a DC resistance on the same wire gives

Equation 13: AC to DC Resistance Ratio

$$\frac{R_{l,\mathrm{AC}}}{R_{l,\mathrm{DC}}} = \frac{A}{A_{eff}}$$

The effective are is

Equation 14: Effective Area

$$A_{eff} = \pi r^2 - \pi\left(r - \delta\right)^2$$

Example 3

An AWG 12 copper conductor has a diameter of 81 mil. The DC resistance is given by a table of conductors at approximately 5.4 Ω/km at 25°C.[10] Calculate the resistance per kilometer at 10 kHz with no temperature change.

Solution

Equation 13 contains the relationship desired. The skin depth, δ, from Ex. 2 is 26 mil. The radius is 40.5 mil (half the diameter). Rearranging Eq. 13 and substituting Eq. 14 gives

[10] Many sources for such table exists. I use the National Electrical Code or the associated handbook of Ref. [D]. Use the table required by the specifications or the appropriate contract. For example, NASA uses different tables for space applications.

$$\frac{R_{l,\mathrm{AC}}}{R_{l,\mathrm{DC}}} = \frac{A}{A_{eff}}$$

$$R_{l,\mathrm{AC}} = R_{l,\mathrm{DC}}\left(\frac{A}{A_{eff}}\right)$$

$$= R_{l,\mathrm{DC}}\left(\frac{\pi r^2}{\pi r^2 - \pi\left(r - \delta\right)^2}\right) = R_{l,\mathrm{DC}}\left(\frac{\pi r^2}{\pi r^2\left(1 - \left(1 - \frac{\delta}{r}\right)^2\right)}\right) = \left(5.4\,\frac{\Omega}{\mathrm{km}}\right)\left(\frac{1}{1 - \left(1 - \frac{26\ \mathrm{mil}}{40.5\ \mathrm{mil}}\right)^2}\right)$$

$$= 6.2\ \Omega/\mathrm{km}$$

INTERNAL INDUCTANCE

The *internal inductance*, also called the *characteristic inductance,* is that inductance associated with the interior of a solid conductor. That is, the fluxes in the space surrounding the conductor are disregarded. For nonmagnetic materials, the relative permeability, μ_r, is approximately one. The magnetic permeability of a conductor is approximately that of free space; that is, $\mu \approx \mu_0$. The internal inductance per unit length for a conductor is

Equation 15: Internal Inductance per unit Length

$$L_{l,\mathrm{int}} = \frac{\mu_0}{8\pi} = 0.5 \times 10^{-7}\ \mathrm{H/m}$$

For a single-phase system, two conductors are present, with current flow in opposite directions, and the total internal inductance is twice the value given in Eq. 15. The *single-phase internal inductance* is

Equation 16: Single Phase Internal Inductance

$$L_{l,\mathrm{int}} = \frac{\mu_0}{4\pi} = 1.0 \times 10^{-7}\ \mathrm{H/m}$$

EXTERNAL INDUCTANCE

The *external inductance* is that inductance associated with the exterior of a solid conductor. That is, the fluxes in the space surrounding the conductor store the inductive energy. For air, the relative permeability, μ_r, is approximately one. The magnetic permeability of air is approximately that of free space, that is, $\mu \approx \mu_0$. The external inductance per unit length for a single conductor is given by

Equation 17: External Inductance

$$L_{l,ext} = \frac{\mu_0}{2\pi}\ln\frac{D}{r} = \left(2\times 10^{-7}\right)\ln\frac{D}{r} \quad \text{[In H/m]}$$

The term D is the distance between the centers of the conductors. The term r is the radius of the conductor. For a single-phase system, two conductors are present, with current flow in opposite directions. The total external inductance is twice the value given in Eq. 17. The *single-phase external inductance* is

Equation 18: Single-Phase External Inductance

$$L_{l,ext} = \frac{\mu_0}{\pi}\ln\frac{D}{r} = \left(4\times 10^{-7}\right)\ln\frac{D}{r} \quad \text{[In H/m]}$$

SINGLE-PHASE INDUCTANCE

The total inductance of a single-phase system, which consists of two conductors, is the sum of the internal and external inductances.[11]

Equation 19: Total Inductance Single-Phase System

$$L_l = L_{l,\text{int}} + L_{l,\text{ext}} = \left(\frac{\mu_0}{4\pi}\right)\left(1 + 4\ln\frac{D}{r}\right) \quad \text{[In H/m]}$$

[11] The equations used in this text assume that $D \gg r$ and that r is identical in the two conductors, which is generally the case in power transmission lines.

The first term of Eq. 19 represents the internal inductance of a solid conductor. The second term represents the inductance caused by external fluxes. Equation 19 is simplified using the concept of the *geometric mean radius*[12] (GMR), defined as

Equation 20: Geometric Mean Radius

$$\text{GMR} = re^{-1/4}$$

Substituting the geometric mean radius for the radius in Eq. 19 gives the following simplified form for the total single-phase inductance per unit length.

Equation 21: Single-Phase Inductance per unit Length using GMR

$$L_l = \left(\frac{\mu_0}{\pi}\right)\left(\ln\frac{D}{\text{GMR}}\right) = 4\times 10^{-7}\left(\ln\frac{D}{\text{GMR}}\right) \quad \text{[In H/m]}$$

Equation 21 is equivalent to Eq. 19 but represents a different model of the conductor. Specifically, Eq. 21 represents an equivalent thin-walled, hollow conductor of radius GMR with no internal flux linkage and, therefore, no internal inductance.

Example 4

The wire in Ex. 3 is used in a single-phase system operating at 10×10^3 Hz with the lines spaced 0.5 m apart.

Ignoring shunt effects, what is the impedance per unit length of line?

Solution

Ignoring the shunt conductance and capacitance gives impedance per unit length as

$$Z_l = R_l + jX_{L,l}$$

[12] The geometric mean radius is sometimes called the *self-geometric mean distance* (GMD).

The resistance per unit length was determined in Ex. 3 to be 6.2×10^{-3} Ω/m. The inductive reactance per unit length is

$$X_{L,l} = 2\pi f L_l$$

The only unknown is the inductance. Using Eq. 19 and Eq. 20, and noting that a radius of 40.5 mil is 1.029×10^{-3} m, gives

$$\begin{aligned} L_l &= \left(4 \times 10^{-7}\ \frac{\text{H}}{\text{m}}\right)\left(\ln \frac{D}{\text{GMR}}\right) = \left(4 \times 10^{-7}\ \frac{\text{H}}{\text{m}}\right)\left(\ln \frac{D}{re^{-1/4}}\right) \\ &= \left(4 \times 10^{-7}\ \frac{\text{H}}{\text{m}}\right)\left(\ln \frac{0.5\ \text{m}}{\left(1.029 \times 10^{-3}\ \text{m}\right)\left(e^{-1/4}\right)}\right) \\ &= 2.57 \times 10^{-6}\ \text{H/m} \end{aligned}$$

The reactance per unit length is

$$\begin{aligned} X_{L,l} &= 2\pi f L_l = (2\pi)\left(10 \times 10^{3}\ \text{Hz}\right)\left(2.57 \times 10^{-6}\ \text{H/m}\right) \\ &= 0.161\ \Omega\text{/m} \end{aligned}$$

The impedance per unit length is

$$\begin{aligned} Z_l &= R_l + jX_{L,l} \\ &= 6.2 \times 10^{-3}\ \Omega\text{/m} + j0.161\ \Omega\text{/m} \end{aligned}$$

The inductive reactance has an effect approximately four times greater than the resistance.

SINGLE-PHASE CAPACITANCE

The shunt capacitance is the capacitance between the solid conductors. Power transmission line conductors are normally separated by air. For air, the relative permittivity, ε_r, is approximately one. Therefore, the permittivity of air is approximately that of free space; that is, $\varepsilon \approx \varepsilon_0$. The capacitance of a single-phase system, which consists of two conductors, is

Equation 22: Single-Phase Capacitance per unit Length

$$C_l = \frac{\pi \varepsilon_0}{\ln \frac{D}{r}} = \frac{2.78 \times 10^{-11}}{\ln \frac{D}{r}} \quad \text{[In F/m]}$$

The term D represents the distance between the centers of the conductors. The term r is the radius of the conductors. For a single-phase system, two conductors are present, with current flow in opposite directions.

Example 5

A copper AWG 20 wire with a radius of 16 mil is used in a single-phase system at 10×10^3 Hz, with the conductors 0.5 m apart.

What is the capacitive reactance per unit length?

Solution

The capacitive reactance per unit length is

$$X_{C,l} = \frac{1}{2\pi f C_l}$$

The only unknown is the capacitance per unit length. Noting that AWG 20 has a radius of 16 mil or 4.064×10^{-4} m. The capacitance per unit length is

$$C_L = \frac{\pi \epsilon_0}{\ln \frac{D}{r}} = \frac{\pi 8.854 \times 10^{-12}\ \frac{\text{F}}{\text{m}}}{\ln\left(\frac{0.5\ \text{m}}{4.064 \times 10^{-4}\ \text{m}}\right)}$$
$$= 3.91 \times 10^{-12}\ \text{F/m}$$

Substitute into the original equation to find the capacitive reactance.

$$X_C = \frac{1}{2\pi f C} = \frac{1}{(2\pi)(10 \times 10^3\ \text{Hz})(3.91 \times 10^{-12}\ \text{F})}$$
$$= 4.07 \times 10^6\ \Omega$$

Shifting to per unit length gives

$$X_{C,l} = \frac{1}{2\pi f C_l}$$
$$= 4.07 \times 10^6\ \Omega/\text{m}$$

The capacitive reactance has a much larger effect on the per-unit reactance of the line than does the inductive reactance.

THREE-PHASE TRANSMISSION

In three-phase, three-wire, or four-wire balanced transmission, the sum of the instantaneous currents is zero, and the sum of the instantaneous voltages is zero.[13] The conductors are assumed to be equilaterally spaced at distance *D*. When the conductors are not symmetrically arranged, as is often the case, formulas used for the inductance and capacitance are still valid if the equivalent distance, D_e, is substituted for the distance (see Fig. 2(a) and Fig. 2(b)).[14]

---

[13] The three-wire or four-wire balanced assumption allows simplifications in the derivations of the equations in this section.

[14] The equivalent spacing is sometimes called the *mutual geometric mean distance* (D_m or GMD).

Equation 23: Equivalent Spacing

$$D_e = \sqrt[3]{D_{\mathrm{ab}} D_{bd} D_{\mathrm{ca}}}$$

Nonsymmetrical spacing causes electrostatic and electromagnetic imbalance among the lines, resulting in unequal phase voltages and currents. In large commercial power transmission systems, the effect is minimal because of the balancing effect of the rotating loads. The effect can be minimized by *transposition* of the lines as illustrated in Fig. 2(c). Transposing the lines improves reliability by minimizing the current induced in the event of a fault, reduces power losses, and reduces interference with nearby telecommunications lines.

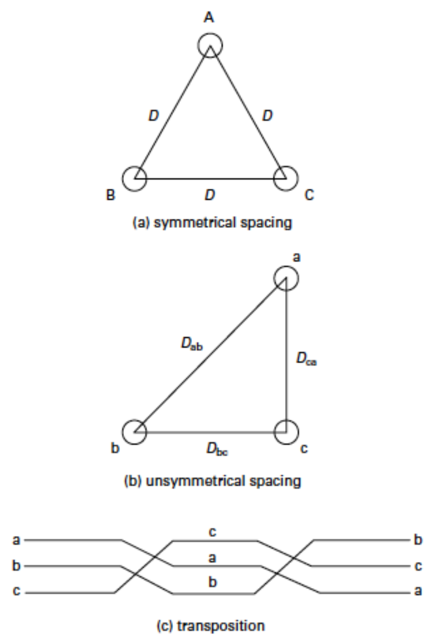


Figure 2: Transmission Line Spacing

The per-phase inductance per unit length of a three-phase transmission line is as follows.

Equation 24: Three-Phase Inductance per unit Length

$$L_l = \frac{\mu_0}{2\pi} \ln \frac{D_e}{\mathrm{GMR}} = \left(2 \times 10^{-7}\right) \ln \frac{D_e}{\mathrm{GMR}} \quad \left[\text{In H/m}\right]$$

Mutual inductance, M, also exists, but for symmetrically spaced conductors the total is zero. This equation is similar to Eq. 21 for the total inductance of a single-phase system. Specifically, it is one-half the single-phase value. *Per-phase values are equivalent to phase-to-neutral values*.

The per-phase capacitance per unit length of a three-phase transmission line is as follows.

Equation 25: Three-Phase Capacitance per unit Length

$$C_l = \frac{2\pi\epsilon_0}{\ln\frac{D_e}{r}} = \frac{5.56\times10^{-11}}{\ln\frac{D_e}{r}} \quad \left[\text{In F/m}\right]$$

This equation is similar to Eq. 22 for the total capacitance of a single-phase system. Specifically, it is twice the single-phase value. Per-phase values are equivalent to phase-to-neutral values.

POWER TRANSMISSION LINES

Power transmission lines normally consist of aluminum strands with a steel core or all-aluminum alloy wires. The lines consist of either three or four conductors. The voltage drop in such lines can be calculated using the principles discussed earlier and the following basic formula.

Equation 26: Transmission Line Voltage

$$\mathbf{V}_l = \mathbf{I}R_l + j\mathbf{I}X_l = \mathbf{I}\mathbf{Z}_l$$

The actual calculation is complicated. Frequency and spacing affect the resistance, inductance, and capacitance. The geometric mean radius given in Eq. 20 must be adjusted for stranded conductors, taking into account both the total number and the layering geometry.

Even so, such calculations are unnecessary, because the properties of standard conductors have been tabulated and the conductors given code names for ease of reference. A partial sample table from the *Aluminum Electrical Conductor Handbook* is given in Table A, which uses English Engineering System units and a 1 ft symmetrical spacing between conductors.

Table 3: Bare Aluminum Conductors Steel Reinforced (ACSR) Electrical Properties

code word	size (kcmil)	stranding (Al/St)	number of aluminum layers	resistance: DC[a] 20°C (Ω/mile)	resistance: AC[b,c] (60 Hz) 25°C (Ω/mile)	resistance: AC[b,c] (60 Hz) 50°C (Ω/mile)	resistance: AC[b,c] (60 Hz) 75°C (Ω/mile)	GMR (ft)	phase-to-neutral (60 Hz) reactance at 1 ft spacing: inductive, X_a (Ω/mile)	phase-to-neutral (60 Hz) reactance at 1 ft spacing: x, X_a' (MΩ–mile)
waxwing	266.8	18/1	2	0.3398	0.347	0.382	0.416	0.0197	0.477	0.109
partridge	266.8	26/7	2	0.3364	0.344	0.377	0.411	0.0217	0.465	0.107
ostrich	300	26/7	2	0.2993	0.306	0.336	0.366	0.0230	0.458	0.106
merlin	336.4	18/1	2	0.2693	0.276	0.303	0.330	0.0221	0.463	0.106
linnet	336.4	26/7	2	0.2671	0.273	0.300	0.327	0.0244	0.451	0.104
oriole	336.4	30/7	2	0.2650	0.271	0.297	0.324	0.0255	0.445	0.103
chickadee	397.5	18/1	2	0.2279	0.234	0.257	0.279	0.0240	0.452	0.103
ibis	397.5	26/7	2	0.2260	0.231	0.254	0.277	0.0265	0.441	0.102
lark	397.5	30/7	2	0.2243	0.229	0.252	0.274	0.0277	0.435	0.101
pelican	477	18/1	2	0.1899	0.195	0.214	0.233	0.0263	0.411	0.100

[a]Direct current (DC) resistance is based on 16.946 Ω-cmil/ft (61.2% IACS) at 20°C for nominal aluminum area of the conductors and 129.64 Ω-cmil/ft (8% IACS) at 20°C for the nominal steel area, with standard increments for stranding. ASTM B 232.

[b]Alternating current (AC) resistance is based on the resistance corrected for temperature using 0.00404 as the temperature coefficient of resistivity per °C for aluminum and 0.0029 per °C for steel and for skin effect.

[c]The effective AC resistance of 3-layer ACSR increases with current density due to core magnetization.

The formula for inductive reactance per mile used in Table 3 is

Equation 27: Inductive Reactance Transmission Cable

$$X_L = \left(2.022\times 10^{-3}\right) f \ln\frac{D}{\text{GMR}} \quad \left(\text{In W/mi}\right)$$

For line-to-line spacings other than 1 ft, the correction factor given by Eq. 28 must be applied.[15]

Equation 28: Inductive Line Spacing Correction

$$K_L = 1 + \frac{\ln D}{\ln\frac{1}{\text{GMR}}}$$

[15] The GMR used in Eq. 27, not Eq. 20, is the one tabulated. However, Eq. 20 is a good approximation of the tabulated value.

The formula for capacitive reactance used in Table 3 is approximately

Equation 29: Capacitive Reactance Transmission Cable

$$X_C = \frac{1.781}{f} \ln \frac{D}{r} \quad \text{[In MW-mi]}$$

For line-to-line spacings other than 1 ft, the correction factor given by Eq. 30 must be applied.

Equation 30: Capacitive Line Spacing Correction

$$K_C = 1 + \frac{\ln D}{\ln \frac{1}{r}}$$

The values of *D*, GMR, and *r* are in feet when used in Eq. 27 through Eq. 30. Table 3 can be used for single-phase systems as well. The values for inductive and capacitive reactance, Eq. 27 and Eq. 29, double for single-phase systems.

Example 6

A single-phase system uses merlin conductors spaced 10 ft apart from center to center. The diameter for the merlin conductors is 0.057 ft.

What is the total shunt reactance for 10 mi of the system?

Solution

From Table 3 for merlin conductors, the capacitive reactance for a 1 ft spacing is 0.106 MΩ-mi. Since the diameter for the merlin conductors is 0.057 ft, the radius is 0.0285 ft. The correction factor for the 10 ft spacing is

$$K_C = 1 + \frac{\ln D}{\ln \frac{1}{r}} = 1 + \frac{\ln 10}{\ln \frac{1}{0.0285}} = 1.647$$

The capacitive reactance for a single conductor is

$$\begin{aligned} X_{C,\text{actual}} &= X_{C,\text{table}} K_C = (0.106 \text{ MW-mi})(1.647) \\ &= 0.1746 \text{ MW-mi} \end{aligned}$$

This is a single-phase system. That is, two conductors are present. The total capacitive reactance is

$$X_{c,\text{total}} = (2)(0.1746 \text{ MW-mi}) = 0.3492 \text{ MW-mi}$$

For a 10 mi portion of the system, the capacitive reactance is

$$X_C = \frac{0.3492 \text{ MW-mi}}{10 \text{ mi}} = 0.03492 \text{ MW}$$

TRANSMISSION LINE REPRESENTATION

Analysis of transmission lines is accomplished by approximating the line with distributed parameters. The line is represented as a series-parallel combination of electrical components applicable to the length of the transmission system.[16] Three-phase transmission systems are operated in as balanced a configuration as possible, but are seldom equilaterally spaced. Transposition may alleviate this imbalance, but it is seldom used.

Nevertheless, calculations assume equilateral spacing and transposition. Calculated values closely approximate actual values if the equivalent spacing, D_e, of Eq. 23 is used in place of the actual spacing. Given these assumptions, balanced three-phase systems can be analyzed on a per-phase

[16] The shunt conductance is normally insignificant and is excluded from the models used.

basis using a hypothetical neutral, which contributes no resistance, inductance, or capacitance. Lines with significant imbalances must be analyzed using symmetrical components.

Design of all transmission lines, regardless of length, must account for the voltage regulation and efficiency of power transmission, η_P. In terms of a *sending end* (subscript *S*), or source, and a *receiving end* (subscript *R*), or load, these quantities are defined by Eq. 31 and Eq. 32.

Equation 31: Transmission Line Voltage Regulation

$$\mathrm{VR} = \frac{\left|V_{R,\mathrm{nl}}\right| - \left|V_{R,\mathrm{fl}}\right|}{\left|V_{R,\mathrm{fl}}\right|}$$

Equation 32: Transmission Line Efficiency

$$h_P = \frac{P_R}{P_S}$$

Any transmission line may be represented as a two-port network as shown in Fig. 3.

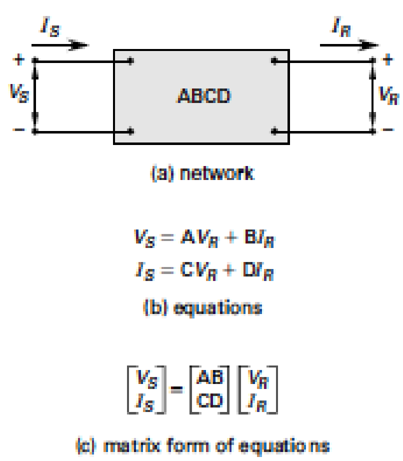


Figure 3: Transmission Line Two-Port Network

The ABCD parameters are sometimes called generalized circuit constants and, in general, are complex. (Though complex, common practice is to not show them in complex notation. That is, they are not shown in bold.) The ABCD parameters for the various transmission lines are given in Table 4.

Table 4: Per-Phase ABCD Constants for Transmission Lines

transmission line length	equivalent circuit	A	B	C	D
short < 80 km (50 mi)	series impedance Fig. 28.4	1	Z	0	1
medium 80–240 km (50–150 mi)	nominal-T Fig. 28.5(a)	$1+\frac{1}{2}YZ$	$Z\left(1+\frac{1}{4}YZ\right)$	Y	$1+\frac{1}{2}YZ$
medium 80–240 km (50–150 mi)	nominal-π Fig. 28.5(b)	$1+\frac{1}{2}YZ$	Z	$Y\left(1+\frac{1}{4}YZ\right)$	$1+\frac{1}{2}YZ$
long > 240 km (150 mi)	distributed parameters Fig. 28.6	$\cosh \gamma l$	$Z_0 \sinh \gamma l$	$\frac{\sinh \gamma l}{Z_0}$	$\cosh \gamma l$

SHORT TRANSMISSION LINES

Short transmission lines are 60 Hz lines that are less than 80 km (50 mi) long. The shunt reactance is excluded from the model because it is much greater than most load impedances and significantly greater than the line impedance. Only the series resistance and inductance are significant as shown in Fig. 4.

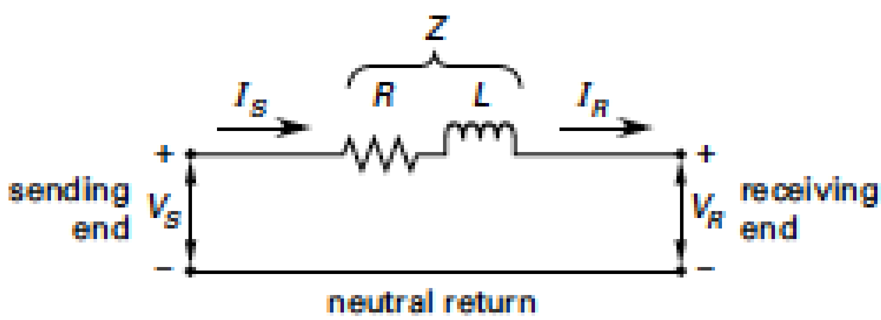


Figure 4: Short Transmission Line Model

The impedance is given by[17]

Equation 33: Impedance Transmission Line

$$Z = R + jX_L$$

Using the ABCD parameters in Table 4 results in Eq. 34 and Eq. 35.

Equation 34: Short Transmission Line Sending End Voltage

$$V_S = V_R + I_R Z$$

Equation 35: Short Transmission Line Sending End Current

$$I_S = I_R$$

Example 7

A 30 mi hawk three-phase 60 Hz transmission line has a 10 ft spacing between conductors. The sending-end voltage is 11 kV per phase. The load draws 200 A per phase at 0.8 pf lagging. The entire system is assumed to operate at 25°C. The properties of a hawk conductor are as follows.

$R_{l,\mathrm{AC}} = 0.193$ W/mi
$\mathrm{X}_{L,l} = 0.430$ W/mi
$\mathrm{GMR} = 0.0290$

What is the voltage regulation in percent?

[17] The quantities represent total amounts but are calculated per unit length, corrected for conductor spacing, and multiplied by the length of the transmission line. The impedance is a per-phase quantity and needs to be multiplied by three to get the line-to-line impedance in a delta-connected load.

Solution

The correction factor for a 10 ft spacing, using the GMR value given, but taken from an expanded version of Table 3, is

$$K_L = 1 + \frac{\ln D}{\ln \frac{1}{\text{GMR}}} = 1 + \frac{\ln 10}{\ln \frac{1}{0.0290}} = 1.6497$$

Note: The units don't appear to work out here. But, the natural logarithm is unitless, as is the common logarithm.

The corrected reactance is

$$X_{L,l(\text{corrected})} = K_L X_{L,l} = (1.6497)\left(0.460 \frac{\text{W}}{\text{mi}}\right) = 0.71 \text{ W/mi}$$

The impedance for 30 mi of line is therefore

$$\begin{aligned} Z &= \left(R_{l,\text{AC}} + jX_{L,l}\right)(30 \text{ mi}) = \left(0.193 \ \frac{\Omega}{\text{mi}} + j0.71 \ \frac{\Omega}{\text{mi}}\right)(30 \text{ mi}) \\ &= 5.79 + j21.3 \ \Omega \\ &= 22.1 \ \Omega\angle 74.8^\circ \end{aligned}$$

The power factor is given as 0.8 lagging; therefore, the current lags the receiving-end voltage by 36.8°. Because this is referenced to the receiving-end voltage, take VR to be the reference at 0°. The current is

$$I_R = 200 \text{ A}\angle -36.8^\circ$$

The voltage drop across the line at full load is

$$\begin{aligned} V_{\text{drop}} &= I_R Z = \left(200 \text{ A}\angle -36.8^\circ\right)\left(22.1 \ \Omega\angle 74.8^\circ\right) \\ &= 4.42 \text{ kV}\angle 38^\circ \end{aligned}$$

Using the relationship for the sending and receiving ends given by Eq. 34 gives

$$\begin{aligned} V_S &= V_R + I_R Z \\ V_R &= V_S - I_R Z \\ &= 11\ \text{kV}\angle\theta - 4.42\ \text{kV}\angle 38^\circ \\ |V_R|\ \text{kV}\angle 0^\circ &= |V_R| + j0 \\ &= (11\cos\theta + j11\sin\theta) - (3.48 + j2.72) \end{aligned}$$

The angle θ is the angle of the sending-end voltage with respect to the receiving-end voltage, which was selected as the reference, which normally is the case.

Because no imaginary portion exists on the left-hand side of the equation, the following must be true.

$$\begin{aligned} 11\sin q - 2.72 &= 0 \\ \sin q &= \frac{2.72}{11} = 0.247 \end{aligned}$$

Use this information to determine the angle and the term $\cos\theta$.

$$\begin{aligned} \theta &= \arcsin 0.247 \\ &= 14.3^\circ \\ \cos\theta &= \cos 14.3^\circ \\ &= 0.969 \end{aligned}$$

Substituting this information gives the receiving-end voltage at full load.

$$\begin{aligned} |V_R|\ \text{kV}\angle 0^\circ &= |V_R| + j0 \\ &= (11\cos\theta + j11\sin\theta) - (3.48 + j2.72) \\ &= \left[(11)(0.969) + j(11)(0.247)\right] - (3.48 + j2.72) \\ &= (10.66 - 3.48) + j(2.72 - 2.72) \\ V_R &= 7.18\ \text{kV}\angle 0^\circ \end{aligned}$$

Note: At this point it's important to note that units weren't carried throughout. One must take care that units stay consistent if this is done.

At a no-load condition, $I_R = 0$ in Eq. 34, and the receiving-end voltage equals the sending-end voltage. That is, $V_{R,\text{nl}} = 11$ kV. The regulation, from Eq. 31, is[18]

$$\text{VR} = \frac{|V_{R,\text{nl}}| - |V_{R,\text{fl}}|}{|V_{R,\text{fl}}|} = \frac{11\ \text{kV} - 7.18\ \text{kV}}{7.18\ \text{kV}} = 0.532 \quad (53.2\%)$$

MEDIUM-LENGTH TRANSMISSION LINES

Medium-length transmission lines are 60 Hz lines between 80 km and 240 km (50 mi and 150 mi) long. The shunt reactance is significant enough to be included. The medium-length line is modeled in one of two ways as shown in Fig. 5.

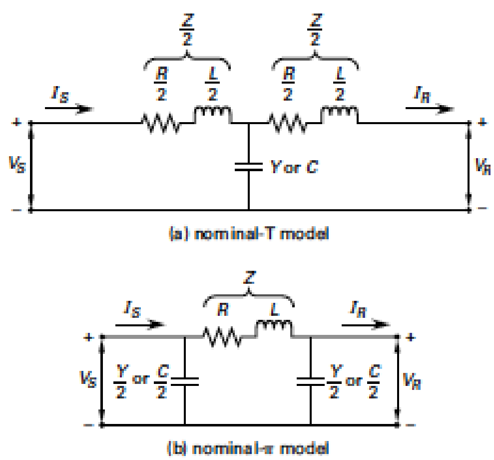


Figure 5: Medium-Length Transmission Line Models

[18] This problem was solved using standard flow in most engineering texts. I have always liked reversing this process: writing what I want to know and then solving for the unknowns. Basically, this entire problem would be accomplished in reverse. Use the method that works for you.

The impedance is[19]

Equation 36: Transmission Line Impedance

$$Z = R + jX_L$$

The admittance is

Equation 37: Transmission Line Admittance

$$Y = jB_C = \frac{j}{X_C} = -\frac{1}{jX_C}$$

Using the ABCD parameters in Table 4 gives Eq. 38 and Eq. 39 *for the T approximation*.

Equation 38: Medium Transmission Line Sending End Voltage [T]

$$V_S = \left(1 + \frac{1}{2}YZ\right)V_R + \left(Z\left(1 + \tfrac{1}{4}YZ\right)\right)I_R$$

Equation 39: Medium Transmission Line Sending End Current [T]

$$I_S = YV_R + \left(1 + \tfrac{1}{2}YZ\right)I_R$$

Using the ABCD parameters in Table 4 gives Eq. 40 and Eq. 41 *for the π approximation*.[20]

Equation 40: Medium Transmission Line Sending End Voltage [π]

$$V_S = \left(1 + \tfrac{1}{2}YZ\right)V_R + ZI_R$$

[19] The quantities represent total amounts but are calculated per unit length, corrected for conductor spacing, and multiplied by the length of the transmission line. Also, the impedance is a per-phase quantity and needs to be multiplied by three to get the line-to-line impedance in a delta-connected load.
[20] The π approximation results in simpler regulation calculations.

Equation 41: Medium Transmission Line Sending End Voltage [π]

$$I_S = \left(Y\left(1+\tfrac{1}{4}YZ\right)\right)V_R + \left(1+\tfrac{1}{2}YZ\right)I_R$$

LONG TRANSMISSION LINES

Long transmission lines are 60 Hz lines greater than 240 km (150 mi) long. All the parameters must be represented as distributed parameters, that is, per unit length, as shown in Fig. 6. The shunt conductance is shown for completeness only and is excluded from most calculations, as will be done here.

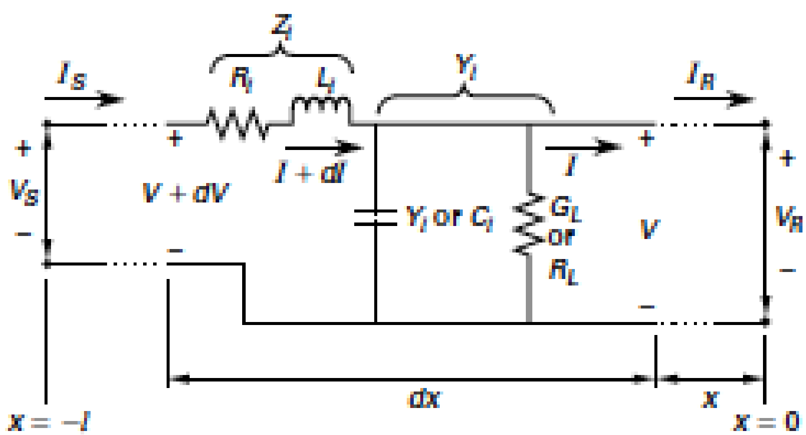


Figure 6: Long Transmission Line Model

The impedance per unit length is[21]

Equation 42: Transmission Line Impedance Per Unit Length

$$Z_l = R_l + jX_l$$

[21] The per unit length impedance and admittance are sometimes written with small letters, *z* and *y*, to emphasize their per-unit qualities.

The admittance per unit length is

Equation 43: Transmission Line Admittance Per Unit Length

$$Y_l = jB_{C,l} = \frac{j}{X_{C,l}} = -\frac{1}{jX_{C,l}} = j\omega C_l$$

The voltage at any point along the line of length *l* in Fig. 6 is given by the wave function as follows.

Equation 44: Wave Equation for Voltage

$$\frac{\partial^2 V}{\partial x^2} = \gamma^2 V$$

The term γ is called the *propagation constant*, measured in radians per meter (mile), and is

Equation 45: Propagation Constant

$$\gamma = \sqrt{Y_l Z_l} = \sqrt{-B_{C,l} X_{L,l} + jYR_l B_{C,l}} = \alpha + j\beta$$

The magnitude of the propagation constant is

Equation 46: Magnitude of Propagation Constant

$$|\gamma| = \sqrt{B_{C,l}}\sqrt[4]{R_l^2 + X_{L,l}^2}$$

The term α is called the *attenuation constant*, measure in nepers[22] per meter (mile), and is

Equation 47: Attenuation Constant

$$\alpha = |\gamma| \cos\left(\tfrac{1}{2}\arctan\frac{-R_l}{X_{L,l}}\right)$$

[22] The neper is a *logarithmic unit for ratios* of measurements of physical field and power quantities.

The term β is called the *phase constant* measure in radian per meter (mile), and is

Equation 48: Phase Constant

$$\beta = |\gamma| \sin\left(\tfrac{1}{2} \arctan \frac{-R_l}{X_{L,l}} \right)$$

A solution to Eq. 44 giving the voltage at any given point along the line is[23]

Equation 49: Voltage Wave Equation Solution

$$V = \tfrac{1}{2} V_R \left(e^{\gamma x} + e^{-\gamma x} \right) + \tfrac{1}{2} I_R Z_0 \left(e^{\gamma x} - e^{-\gamma x} \right) = V_R \cosh \gamma x + I_R Z_0 \sinh \gamma x$$

Similarly, the current at any given point along the line is

Equation 50: Current Wave Equation Solution

$$I = \tfrac{1}{2} I_R \left(e^{\gamma x} + e^{-\gamma x} \right) + \tfrac{1}{2} \frac{V_R}{Z_0} \left(e^{\gamma x} - e^{-\gamma x} \right) = I_R \cosh \gamma x + \left(\frac{V_R}{Z_0} \right) \sinh \gamma x$$

The term Z_0, called the *characteristic impedance*, is

Equation 51: Characteristic Impedance

$$Z_0 = \sqrt{\frac{Z_l}{Y_l}} = \sqrt{\frac{R_l + jX_{L,l}}{jB_{C,l}}} = \sqrt{X_{L,l} X_{C,l} - jX_{C,l} R_l} \quad \text{[In } \Omega\text{]}$$

The magnitude of the characteristic impedance is

Equation 52: Characteristic Impedance Magnitude

$$|Z_0| = \sqrt{X_{C,l}} \sqrt[4]{R_l^2 + X_l^2}$$

[23] The solution assumes a sinusoidal steady-state condition. That is, transmission lines are generally AC steady-state networks. (This is all leading to a nice overall equation that helps one avoid the Smith Chart.)

The angle of the characteristic impedance is

Equation 53: Characteristic Impedance Angle

$$\angle Z_0 = \tfrac{1}{2}\arctan\frac{-R_l}{X_{L,l}}$$

Equation 49 and Eq. 50 represent *traveling waves*. Specifically, the terms with $e^{-\gamma x}$ represent waves traveling in the positive *x*-direction. (The distance x is defined as positive to the right, the direction toward the receiving end, that is, toward the direction power flows.) Terms with $e^{\gamma x}$ represent waves traveling in the negative *x*-direction. The two equations are the value of the two waves superimposed at that point in the line. Because $V = V_S$ (see Eq. 49) at $x = -l$ (see Fig. 6), and consistent with the ABCD parameters in Table 4, the voltage at the sending end is[24]

Equation 54: Sending End Voltage Summary

$$V_S = V_R \cosh \gamma l + I_R Z_0 \sinh \gamma l$$

Because $I = I_S$ (see Eq. 50) at $x = -l$ (see Fig. 6), and consistent with the ABCD parameters in Table 4, the current at the sending end is

Equation 55: Sending End Current Summary

$$I_S = I_R \cosh \gamma l + \frac{V_R}{Z_0} \sinh \gamma l$$

The term *l* in Eq. 51 and Eq. 52 is the transmission line length.

[24] This is likely confusing—easily so. But, look closely at Fig. 6, *x goes from a negative value at the point of generation in the positive direction*, to 0 at the receiving end (the load), which is where the engineer is likely focused. The hyperbolic cosine and sine incorporate the exponentials. The first term is the traveling voltage that made it to the receiving end. The second term is that which exists on the line but is diminished by the resistance of the line.

REFLECTION COEFFICIENT

Energy propagated along a transmission line can be thought of as traveling electromagnetic waves. Loads interact with these waves in a manner that absorbs some of the energy and reflects the remainder. If the transmission line is terminated with the characteristic impedance, Z_0, no power is reflected back to the source or sending end. If the termination impedance is other than Z_0, the power (signal) from the generator (source) will be partially reflected back to the generator. Assuming a steady-state sinusoidal source, the generator's waves will then combine in the transmission line with the reflected waves to form *standing waves*. The *standing wave ratio*, SWR, is the ratio of the maximum to the minimum voltages (currents) encountered along the transmission line. Typically, the SWR is greater than unity. If the terminating impedance matches the characteristic impedance, all the input power provided by the generator is absorbed by the load and the SWR equals one. The standing wave ratios are

Equation 56: Voltage Standing Wave Ratio

$$\text{VSWR} = \frac{V_{\text{max}}}{V_{\text{min}}}$$

Equation 57: Current Standing Wave Ratio

$$\text{ISWR} = \frac{I_{\text{max}}}{I_{\text{min}}}$$

The *reflection coefficient*, Γ, is the ratio of the reflected to the incident electric parameter. The voltage reflection coefficient for the load is

Equation 58: Voltage Reflection Coefficient

$$\Gamma_L = \frac{V_{\text{reflected}}}{V_{\text{incident}}} = \frac{Z_{\text{load}} - Z_0}{Z_{\text{load}} + Z_0}$$

The current reflection coefficient is the negative of the reflection coefficient for the voltage. That is,

Equation 59: Current Reflection Coefficient

$$\Gamma_L = \frac{I_{\text{reflected}}}{I_{\text{incident}}} = \frac{Z_0 - Z_{\text{load}}}{Z_0 + Z_{\text{load}}}$$

The fraction of incident power that is reflected back to the source from the load is Γ^2. The relationship between the reflection coefficient and the standing wave ratio is

Equation 60: Reflection Coefficient vs Standing Wave Ratio

$$\Gamma = \frac{\text{SWR} - 1}{\text{SWR} + 1}$$

TRANSMISSION LINE IMPEDANCE

The equations presented so far assume lossless transmission lines, so that I^2R losses and insulated leakage losses given by G^2V are ignored. In most practical instances, transmission line losses are negligible, so they are commonly ignored. Transmission lines are large in one dimension (the distance or length from source to load) and small in the other two dimensions (the width and height of the conductors and cable configuration). For this reason, the voltage and current waves are directly related to the **E** and **H** fields in the spaces between the conductors. Maxwell's equations can be used to derive the equations for a plane traveling wave.

Equation 61: Curl of Electric Field, Point Form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Equation 62: Curl of Magnetic Field Strength, Point Form

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$$

From Eq. 61 and Eq. 62, as well as Eq. 51 through Eq. 53 for characteristic impedance, the impedance can be found for any position along a long, lossless transmission line with a sinusoidal source. The impedance along the line at position x = −d, where d is a positive distance measured from the receiving end, is given by the following[25] (see Ref. [J] or similar for complete derivation).

Equation 63: Transmission Line Impedance

$$Z(d) = \frac{V(d)}{I(d)} = Z_0\left(\frac{1+\Gamma e^{-j2\beta d}}{1-\Gamma e^{-j2\beta d}}\right) = Z_0\left(\frac{Z_R\cos\beta d + jZ_0\sin\beta d}{Z_0\cos\beta d + jZ_R\sin\beta d}\right)$$

Equation 63 can be used in place of the Smith chart for finding the impedance at a given point along a transmission line. (For Smith chart coordinates, see Ref. [B]).[26]

The term β is the phase constant defined earlier in terms of the attenuation constant (see Eq. 48). β can also be expressed in terms of the velocity of propagation and wavelength.

Equation 64: Phase Constant

$$\beta = \frac{\omega}{\mathrm{v}_{\mathrm{wave}}} = \frac{2\pi}{\lambda}$$

A transmission line that is short-circuited at the receiving end has an impedance of $Z_R = 0\ \Omega$ and a voltage of $V_R = 0$ V. Substituting the receiving-end impedance of 0 Ω into Eq. 63 gives

Equation 65: Short-Circuit Reactance

$$Z_{sc} = jZ_0\tan\beta d$$

Because no power can be dissipated in a short circuit, the short-circuit impedance is purely reactive.

---

[25] Euler's relation is used to change the exponential terms to sinusoidal terms. The $x = 0$ point is at the receiving end of the transmission line and is negative when moving to the left toward a source. Since $x = -d$, Eq. 63 is as shown, but d can be considered a positive distance from the receiving end.
[26] As mentioned, this equation may be used instead of the Smith Chart. The Smith Chart is useful for understanding. Now that calculators rule, one my find using the equation easier—and definitely quicker.

A transmission line that is open-circuited at the receiving end has an impedance of $Z_R = \infty\ \Omega$, and $I_R = 0$ A. Substituting the receiving-end impedance of $\infty\ \Omega$ into Eq. 63 gives

Equation 66: Open-Circuit Reactance

$$Z_{oc} = -jZ_0 \cot \beta d$$

An open-circuited line is capacitive for $d < \lambda/4$ and inductive for $\lambda/4 < d < \lambda/2$, after which the pattern repeats. So, a short-circuited line, which is inductive when $d < \lambda/4$, is equivalent to an open-circuited line that is longer than the short-circuited line by $\lambda/4$ in the range $\lambda/4 < d < \lambda/2$. Figure 7 illustrates this concept.

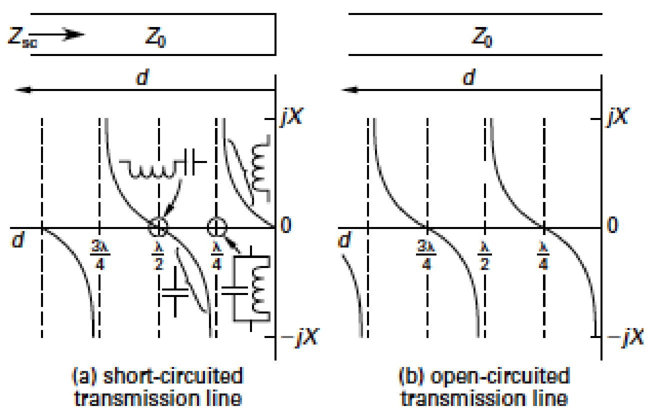


Figure 7: Input Impedance of short-Circuited and Open-Circuited Transmission Lines

The value of d can be varied to achieve any desired results. Open-circuited and short-circuited stubs can be used on transmission lines to ensure that the impedance of the source-line combination matches the impedance of the receiving-end load. A *stub* is a length of transmission wire that is connected only at one end, with the free end left either open-circuited or short-circuited. The stubs reduce or eliminate reflections and ensure maximum power transfer. The characteristic impedance of a finite transmission line with open-circuit and short-circuit terminations is given by

Equation 67: Characteristic Impedance using Short- and Open-Circuit Terminations

$$Z_0 = \sqrt{Z_{sc} Z_{oc}}$$

HIGH-FREQUENCY TRANSMISSION LINES

At high frequencies, approximately 1 MHz and higher, wavelengths are shorter and even a few feet of line are treated as a long transmission line. The resistance is normally negligible, specifically, R ωL and G ωC. Setting the resistance, R, equal to zero simplifies the governing equations. The impedance per unit length is[27]

Equation 68: Impedance High-Frequency Line

$$Z_l = jX_{L,l} = j\omega L_l$$

High-frequency transmission lines use the same admittance per unit length equation as long transmission lines (Eq. 43).[28]

The propagation constant, γ, in units of radians per unit length is

Equation 69: Propagation Constant High-Frequency Line

$$\gamma = \sqrt{Y_l Z_l} = \sqrt{\left(j\omega C_l\right)\left(j\omega L_l\right)} = j\omega\sqrt{L_l C_l} = j\beta$$

The term β in Eq. 69 is the phase constant. The phase velocity, v_{phase}, and the wavelength are

Equation 70: Phase Velocity High-Frequency Line

$$v_{phase} = \frac{1}{\sqrt{L_l C_l}}$$

Equation 71: Wavelength High-Frequency Line

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{L_l C_l}}$$

---

[27] The per-unit-length impedance and admittance are sometimes written with lowercase letters, z and y, to emphasize their per-unit qualities. In this course, the lowercase letters are used for normalized quantities.

[28] In fact, all the equations in this section are applicable to transmission lines. The parameters used in high frequency lines may vary from those normally used.

The characteristic impedance, Z_0, is[29]

Equation 72: Characteristic Impedance High-Frequency Line

$$Z_0 = \sqrt{\frac{Z_l}{Y_l}} = \sqrt{\frac{jwL_l}{jwC_l}} = \sqrt{\frac{L_l}{C_l}}$$

Equation 49, Eq. 50, Eq. 54, and Eq. 55 for long power transmission lines are applicable to high frequency lines when the values from Eq. 43, Eq. 68, Eq. 69, and Eq. 72 are substituted.

At high frequencies, however, circuits are designed to minimize reflected waves. This is accomplished by using capacitive compensation to make the load impedance appear to be the characteristic impedance. The expressions for the sending voltage and current are specified in terms of the reflection coefficient, Γ. The voltage traveling in the positive *x* direction toward the receiving end is associated with

Equation 73: Positive Traveling Wave High-Frequency Line

$$V^+ = \tfrac{1}{2}V_R + \tfrac{1}{2}Z_0 I_R$$

Equation 73 gives one of the constants from the solution of the wave equation, Eq. 44. The other constant is V^-. Both can be determined from the solution to the wave equation given by Eq. 49. The positive *x*-direction is defined as the direction of power flow, that is, from the sending end to the receiving end.

Equation 74: Negative Traveling Wave High-Frequency Line

$$\frac{\partial^2 V}{\partial x^2} = \gamma^2 V$$

The terms cosh $j\beta x = \cos\beta x$ and sinh $j\beta x = j\sin\beta x$, when combined with Eq. 73, allow the sending-end voltage and current to be written in terms of the reflection coefficient.

[29] The characteristic impedance is the ratio of the voltage to the current on an infinite transmission line. It can be called the *characteristic resistance*, R_0, because the line is lossless. That is, Z_l and Y_l are purely reactive, making Z_0 purely resistive.

Equation 75: Sending End Voltage High Frequency Line

$$V_S = V^+\left(e^{j\beta l} + \Gamma e^{-j\beta l}\right) = V^+\left(\left(1+\Gamma\right)\cos\beta l + j\left(1-\Gamma\right)\sin\beta l\right)$$

Equation 76: Sending End Current High Frequency Line

$$I_S = \left(\frac{V^+}{Z_0}\right)\left(e^{j\beta l} - \Gamma e^{-j\beta l}\right) = \left(\frac{V^+}{Z_0}\right)\left(\left(1-\Gamma\right)\cos\beta l + j\left(1+\Gamma\right)\sin\beta l\right)$$

The input impedance is V_S/I_S and, in terms of the reflection coefficient, is given by

Equation 77: Input Impedance using Reflection Coefficient

$$Z_{in} = Z_o\left(\frac{\left(1+\Gamma\right)\cos\beta l + j\left(1-\Gamma\right)\sin\beta l}{\left(1-\Gamma\right)\cos\beta l + j\left(1+\Gamma\right)\sin\beta l}\right)$$

The reflection coefficient is defined in Eq. 58. Using this definition, the input impedance can be written in terms of the load impedance, as in Eq. 78.[30]

Equation 78: Input Impedance using know Impedances

$$Z_{\text{in}} = Z_0\left(\frac{Z_{\text{load}}\cos\beta l + jZ_0\sin\beta l}{Z_0\cos\beta l + jZ_{\text{load}}\sin\beta l}\right)$$

The voltage at any point on a transmission line is a function of the distance from the load because there are two waves traveling on any line, one incident to the load and one reflected from the load. The standing wave ratios defined by Eq. 56 and Eq. 57 measure the difference between the maximum rms voltage (current) and the minimum rms voltage (current) of the two waves. (The standing wave ratio is a physical variable that can be directly measured.) Where the magnitudes of the waves add, the voltage (current) is at a maximum. Where the magnitudes of the waves subtract, the voltage (current) is at a minimum. The reflection coefficient is defined in terms of the standing wave ratios by Eq. 60.

[30] This is the equation most used, and again, may substitute for the Smith Chart.

Rearranging Eq. 60 to define the voltage standing wave ratio, VSWR, in terms of the reflection coefficient gives

Equation 79: VSWR

$$\text{VSWR} = \frac{|V^+| + |V^-|}{|V^+| - |V^-|} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

The VSWR is related to the minimum and maximum impedance by Eq. 80 and Eq. 81.

Equation 80: Maximum Impedance

$$Z_{\text{max}} = Z_0(\text{VSWR})$$

Equation 81: Minimum Impedance

$$Z_{\text{min}} = \frac{Z_0}{\text{VSWR}}$$

Example 8

A certain transmission line has a characteristic impedance of 50 Ω and a terminating resistance of 100 Ω.

What is the reflection coefficient?

Solution

The reflection coefficient is defined by Eq. 58 as

$$\Gamma_L = \frac{V_{\text{reflected}}}{V_{\text{incident}}} = \frac{Z_{\text{load}} - Z_0}{Z_{\text{load}} + Z_0}$$

Substitute the given values.

$$\Gamma_L = \frac{Z_{\text{load}} - Z_0}{Z_{\text{load}} + Z_0} = \frac{100\ \Omega - 50\ \Omega}{100\ \Omega + 50\ \Omega} = 0.333$$

Example 9

Consider the transmission line described in Ex. 8.

What is the input impedance if the line is an even number of wavelengths long?

Solution

The transmission line is restricted by the problem statement to be an even number of wavelengths for the frequency carried. In mathematical terms, with n as an integer, this is stated as

$$l = n\lambda = \frac{2\pi n}{\beta}$$

Equation 71 was used to change the wavelength, λ, into a function of the phase constant, β. Rearranging gives

$$\beta l = 2\pi n$$

This wavelength restriction causes the sinusoidal terms of the input impedance equation, Eq. 78, to be

$$\cos \beta l = 1$$
$$\sin \beta l = 0$$

Substituting these values into Eq. 78 gives

$$
\begin{aligned}
Z_{\text{in}} &= Z_0\left(\frac{Z_{\text{load}}}{Z_0}\right) \\
&= Z_{\text{load}} = 100\ \Omega
\end{aligned}
$$

Example 10

Consider the transmission line in Ex. 8.

What is the voltage standing wave ratio?

Solution

The VSWR can be found by substituting the calculated value of the reflection coefficient (determined in Ex. 8) into Eq. 79.

$$
\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.333}{1-0.333} = 2
$$

SUMMARY

Many equations are in this course. They are there as part of a derivation of more important overall traveling wave equations (the Smith Chart replacements), or to define parameters or constants used in said equations. As in most cases, the derivations once understood and accepted as valid, may be dropped and the focus can shift to the final result and their use and application. Most of the properties of conductors are now tabulated and the equations programmed into the associated software. The engineer should understand the equations used by the software and have a “feel” for anticipated results. In such endeavors, all the best.

REFERENCES

A. Camara, John A. *Electrical Engineering Reference Manual.* Belmont, CA: PPI, 2009.
B. Camara, John A. *PE Power Reference Manual*. Belmont, CA: PPI (Kaplan), 2021.
C. Marne, David J., and John A. Palmer. *National Electrical Safety Code®* (NESC®) *2023 Handbook.* New York: McGraw Hill, 2023.
D. Earley, Mark, ed. *NFPA 70, National Electrical Code Handbook*. Quincy, Massachusetts: NFPA, 2020.
E. IEEE 315-1975. Graphic Symbols for Electrical and Electronics Diagrams. New York: IEEE, approved 1975, reaffirmed 1993.
F. IEEE 280-2021. IEEE Standard Letter Symbols for Quantities Used in Electrical Science and Electrical Engineering. New York: IEEE.
G. Grainger, John J., and William Stevenson, Jr. *Power System Analysis*. New York, McGraw Hill, 1994.
H. Grigsby, Leonard L., ed. *Electric Power Generation, Transmission, and Distribution.* Boca, Raton, FL: CRC Press, 2012
I. Wikipedia, The Free Encyclopedia (2024). “Per-Unit System”. https://en.wikipedia.org/wiki /Per-unit_system (accessed FEB 2024).
J. Plonus, Martin A. *Applied Electromagnetics.* New York, McGraw Hill, 1978.[31]

[31] This book, as are many others, are old—like this author. But, the principles of electrical engineering remain. So, accuracy and understandability are the most important.

Appendix A: Equivalent Units Of Derived And Common SI Units

Symbol	Equivalent Units			
A	C/s	W/V	V/W	J/(s × V)
C	A × s	J/V	(N × m)/V	V × F
F	C/V	C^2/J	s/W	(A × s)/V
F/m	C/(V × m)	C^2/(J × m)	C^2/(N × m^2)	s/(Ω × m)
H	W/A	(V × s)/A	Ω × s	(T × m^2)/A
Hz	1/s	s^{-1}	cycles/s	radians/(2p × s)
J	N × m	V × C	W × s	(kg × m^2)/s^2
m^2/s^2	J/kg	(N × m)/kg	(V × C)/kg	(C × m^2)/(A × s^3)
N	J/m	(V × C)/m	(W × C)/(A × m)	(kg × m)/s^2
N/A^2	Wb/(N × m^2)	(V × s)/(N × m^2)	T/N	1/(A × m)
Pa	N/m^2	J/m^3	(W × s)/m^3	kg/(m × s^2)
W	V/A	W/A^2	V^2/W	(kg × m^2)/(A^2 × s^3)
S	A/V	1/W	A^2/W	(A^2 × s^3)/(kg × m^2)
T	Wb/m^2	N/(A × m)	(N × s)/(C × m)	kg/(A × s^2)
V	J/C	W/A	C/F	(kg × m^2)/(A × s^3)
V/m	N/C	W/(A × m)	J/(A × m × s)	(kg × m)/(A × s^3)
W	J/s	V × A	V^2/W	(kg × m^2)/s^3
Wb	V × s	H × A	T/m^2	(kg × m^2)/(A × s^2)

Appendix B: Physical Constants[1]

Quantity	Symbol	US Customary	SI Units
Charge			
electron	e		-1.6022×10^{-19} C
proton	p		$+1.6022 \times 10^{-19}$ C
Density			
air [STP][32°F, (0°C)]		0.0805 lbm/ft^3	1.29 kg/m^3
air [70°F, (20°C), 1 atm]		0.0749 lbm/ft^3	1.20 kg/m^3
sea water		64 lbm/ft^3	1025 kg/m^3
water [mean]		62.4 lbm/ft^3	1000 kg/m^3
Distance			
Earth radius[2]	⊕	2.09×10^{7} ft	6.370×10^{6} m
Earth-Moon separation[2]	⊕☾	1.26×10^{9} ft	3.84×10^{8} m
Earth-Sun separtion[2]	⊕⊙	4.89×10^{11} ft	1.49×10^{11} m
Moon radius[2]	☾	5.71×10^{6} ft	1.74×10^{6} m
Sun radius[2]	⊙	2.28×10^{9} ft	6.96×10^{8} m
first Bohr radius	a_0	1.736×10^{-10} ft	5.292×10^{-11} m
Gravitational Acceleration			
Earth [mean]	g	32.174 (32.2) ft/sec^2	9.8067 (9.81) m/s^2
Mass			
atomic mass unit	$\propto$ or $m_{\propto}$ $\frac{1}{12}m\left(^{12}C\right)$	3.66×10^{-27} lbm	1.6606×10^{27} kg or 10^{-3} kg mol^{-1} / N_A
Earth[2]	⊕	4.11×10^{23} slugs	6.00×10^{24} kg
Earth [customary U.S.][2]	⊕	1.32×10^{25} lbm	-
Moon[2]	☾	1.623×10^{23} lbm	7.36×10^{22} kg
Sun[2]	⊙	4.387×10^{30} lbm	1.99×10^{30} kg
electron rest mass	m_e	2.008×10^{-30} lbm	9.109×10^{-31} kg
neutron rest mass	m_n	3.693×10^{-27} lbm	1.675×10^{-27} kg
proton rest mass	m_p	3.688×10^{-27} lbm	1.672×10^{-27} kg
Pressure			
atmospheric		14.696 (14.7) lbf/in^2	1.0133×10^{5} Pa
Temperature			
standard		32 F (492 R)	0 C (273 K)
absolute zero		- 459.67 F (0 R)	- 273.16 C (0 K)
Velocity[3]			
Earth escape		3.67×10^{4} ft/sec	1.12×10^{4} m/s
light (vacuum)	c, c_0	9.84×10^{8} ft/sec	2.9979 (3.00) $\times 10^{8}$ m/s
sound [air, STP]	a	1090 ft/sec	331 m/s

Table Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/.
2. Symbols shown for the solar system are those used by NASA. See https://science.nasa.gov/resource/solar-system-symbols/.
3. Velocity technically is a vector. It has direction.

Appendix C: Fundamental Constants

Quantity	Symbols	US Customary	SI Units
Avogadro's number	N_A, L		6.022×10^{23} mol^{-1}
Bohr magneton	μ_B		9.2732×10^{-24} J/T
Boltzmann constant	k	5.65×10^{-24} ft-lbf / R	1.3805×10^{-23} J/T
electron volt: $\left(\frac{e}{C}\right)$ J	eV		1.602×10^{-19} J
Faraday constant, $N_A e$	F		96485 C/mol
fine structure constant, inverse α^{-1}	α α^{-1}		7.297×10^{-3} ($\approx$1/137) 137.035
gravitational constant	g_c	32.174 lbm-ft/lbf-sec^2	
Newtonian gravitational constant	G	3.44×10^{-8} ft^4 / lbf-sec^4	6.672×10^{-11} N$\cdot$m^2 / kg^2
nuclear magneton	μ_N		5.050×10^{-27} J/T
permeability of a vacuum	μ_0		1.2566×10^{-6} N/A^2 (H/m)
permittivity of a vacuum, electric constant $1/\mu_0 c^2$	ε_0		8.854×10^{-12} C^2 / N$\cdot$m^2 (F/m)
Planck's constant	h		6.6256×10^{-34} J$\cdot$s
Planck's constant: h/2π			1.0546×10^{-34} J$\cdot$s
Rydberg constant	R_∞		1.097×10^{7} m^{-1}
specific gas constant, air	R	53.3 ft-lbf/lbm- R	287 J/kg$\cdot$K
Stefan-Boltzmann constant		1.71×10^{-9} BTU/ft^2-hr-°R^4	5.670×10^{-8} W/m$^2\cdot$K^4
triple point, water		32.02 F, 0.0888 psia	0.01109 C, 0.6123 kPa
universal gas constant	R*	1545 ft-lbf/lbmol- R 1.986 BTU/lbmol- R	8314 J/kmol$\cdot$K

Table Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/. The unit in Volume of “lbmol” is an actual unit, not a misspelling.

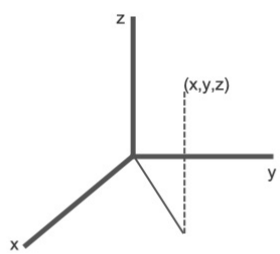
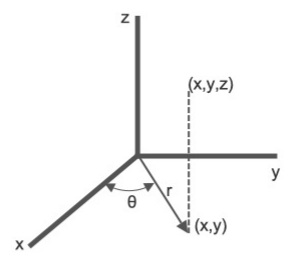
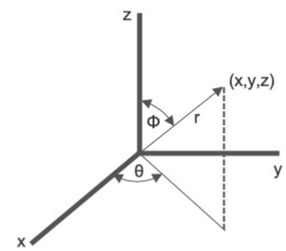
Appendix D: Mathematical Constants

Quantity	Symbol	Value
Archimedes' constant (pi)	p	3.1415926536
base of natural logs	*e*	2.7182818285
Euler's constant	*C* or t	0.5772156649

Appendix E: The Greek Alphabet

A	*a*	alpha	N	*u*	nu
B	*b*	beta	X	x	xi
G	*g*	gamma	O	*o*	omicron
D	*d*	delta	P	*p*	pi
E	*e*	epsilon	R	*r*	rho
Z	z	zeta	S	*s*	sigma
H	*h*	eta	T	*t*	tau
Q	*q*	theta	ϒ	*u*	upsilon
I	i	iota	F	*f*	phi
K	*k*	kappa	C	*c*	chi
L	*l*	lambda	Y	*y*	psi
M	*m*	mu	W	*w*	omega

Appendix F: Coordinate Systems & Related Operations

Mathematical Operations	Rectangular Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Rectangular Coordinants	 $x = x$ $y = y$ $z = z$	 $x = r\cos q$ $y = r\sin q$ $z = z$	 $x = r\sin f\cos q$ $y = r\sin f\sin q$ $z = r\cos f$
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial q}\boldsymbol{\theta} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial f}\boldsymbol{\phi} + \frac{1}{r\sin q}\frac{\partial f}{\partial q}\boldsymbol{\theta}$
Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_q}{\partial q} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin f}\frac{\partial (A_f \sin f)}{\partial f} + \frac{1}{r\sin f}\frac{\partial A_q}{\partial q}$
Curl	$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\theta} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial q} & \frac{\partial}{\partial z} \\ A_r & A_q & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r^2\sin q}\mathbf{r} & \frac{1}{r^2\sin q}\boldsymbol{\phi} & \frac{1}{r}q \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial f} & \frac{\partial}{\partial q} \\ A_r & rA_f & rAqA_f \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r}\frac{\partial r}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin f}\frac{\partial}{\partial f}\left(\sin f\frac{\partial f}{\partial f}\right) + \frac{1}{r^2\sin^2 f}\left(\frac{\partial^2 f}{\partial q^2}\right)$