AC Electrical 101+

Part I

Concepts & Single Phase

Fundamentals / Voltage / Current / Impedance / RMS vs Maximum Values / Phasors / Complex Representation / Real Power / Reactive Power / Apparent Power

by

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Course 538

5 PDH (5 Hours)

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Nomenclature[1]

a	phase	-
a	turns ratio	-
a_n, b_n	Fourier coefficients	-
A	ABCD parameter	-
A	area	m^2
B	ABCD parameter	-
B	magnetic flux density	T
B	magnetic flux density	T
B	susceptance	S, Ω^{-1}, or mho
b	phase	-
c	speed of light	m/s
C	capacitance	F
C	ABCD parameter	-
c	phase	-
CF	crest factor	-
D	ABCD parameter	-
D	distance	m
E	electric field strength	V/m
E	energy	J
E	voltage (generated)	V
E	electromotive force	V
f	form factor	-
f	frequency	Hz, s^{-1}, cycles/s
f_{droop}	frequency droop	Hz/kW
G	conductance	S, Ω^{-1}, or mho
GMD	geometric mean distance	m
GMR	geometric mean radius	m
h	specific enthalpy	kJ/kg
i	variable current	A
I	effective (rms) or DC current	A
I	rms phasor current	A

[1] Not all the nomenclature, symbols, or subscripts may be used in this course—but they are related, and may be found when reviewing the references listed for further information. Further, all the nomenclature, symbols, or subscripts will be found in of many electrical courses (on SunCam, PDH Academy, and also in many texts). For guidance on nomenclature, symbols, and electrical graphics: IEEE 280-2021. IEEE Standard Letter Symbols for Quantities Used in Electrical Science and Electrical Engineering. New York: IEEE; and IEEE 315-1975. Graphic Symbols for Electrical and Electronics Diagrams. New York: IEEE, approved 1975, reaffirmed 1993.

I^*	complex conjugate of current	A
K	correction factor	-
K	skin effect ratio	-
l	length	m
L	inductance	H
m	mass	kg
M	mutual inductance	H
N	number of turns	-
n	Steinmetz exponent	-
n_s	synchronous speed	r/min or min^{-1}
p	instantaneous power	W
p	pressure	Pa
P	number of poles	-
P	power	W
pf	power factor	-
pu	per unit	-
Q	reactive power	VAR
Q	heat	J
r	radius	m
R	resistance	Ω
s	specific entropy	$\text{kJ/kg} \cdot \text{K}$
S	apparent power or complex power	VA
SWR	standing wave ratio	-
t	time	s
T	period	s
T	temperature	ºC or K
υ	variable voltage	V
v	velocity (speed)	m/s
v	wind velocity	km/hr
V	constant or rms voltage	V
V	effective or DC voltage	V
V	rms phasor voltage	V
V_{droop}	voltage droop	V/kVAR
VR	voltage regulation	-
W	work	kJ
x	variable	-
X	reactance	Ω

y	admittance per unit length	S/m, $1/\Omega\cdot\text{m}$ Ω^{-1}, or mho/m [$\mho$ / m]
Y	admittance	S, Ω^{-1}, or mho
z	impedance per unit length	Ω/m
Z	complex number	-
Z	impedance	Ω
Z_0	characteristic impedance	Ω

Symbols

α	turns ratio	-
α	attenuation constant	Np/m
α	thermal coefficient of resistance	1/°C
β	phase constant	rad/m
γ	propagation constant	rad/m
Γ	reflection coefficient	-
δ	skin depth	m
Δ	change, final minus initial	-
ε	permittivity	F/m
ε_0	free-space permittivity	8.854×10^{-12} F/m
ε_r	relative permittivity	-
η	efficiency	-
θ	phase angle	rad
κ	coupling coefficient	-
μ	permeability	H/m
μ_o	free-space permeability	1.2566×10^{-6} H/m
μ_r	relative permeability	-
ξ	ratio of radii	-
ρ	resistivity	$\Omega \cdot m$
σ	conductivity	S/m
ϕ	impedance angle or power angle	rad
ϕ	current angle	rad
ϕ	phase angle difference	rad
ϕ_{pf}	power factor angle	rad
ω	armature angular speed	rad/s

Subscripts

ϕ	phase
0	zero sequence
0	characteristic
0	free space (vacuum)
0,o	initial (zero value)
1	positive sequence
1	primary
2	negative sequence
2	secondary
ab	a to b
AC	alternating current
avg, ave	average
bc	b to c
c	controls or critical
c	core
C	capacitive or capacitor
ca	c to a
Cu	copper
d	direct
DC	direct current
e	eddy current
e	equivalent
eff	effective
ext	external
E	generated voltage
f	final / frequency
fl	full load
g or *gen*	generator
h	hysteresis
i	imaginary
int	internal
I	current
l	line

l	per unit length
L	inductor or load
ll	line-to-line
ln	line to neutral
m	motor
m	maximum
m	mutual
max	maximum
n	neutral
nl	no load
O	origin
oc	open circuit
p	peak
p	phase
p	primary
pf	power factor
ps	primary to secondary
pu	per unit
q	quadrature
rms	root mean square
r	real
R	receiving end
R	resistance or resistive
s	secondary
s	source
s	synchronous
S	sending end
sc	short circuit
t	terminal or total
thr	threshold
trans	transmission
V	potential difference between two points
w	wave
Z	impedance

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COURSE INTRODUCTION

The information is primarily from the author's books, Refs. [A] and [B], and focuses on the basics of AC electricity. The NESC Ref. [C] and NEC Ref. [D] though not covered in this course are useful sources for electrical engineers. Information useful in many aspects of electric engineering may be found in [E] and [F] as well as the appendices. Reference [G] has detailed descriptions of analysis techniques. Reference [H] provides detailed engineering review. Reference [I] provides indepth explanation of the per-unit system often used in such engineering. Reference [J] covers many terms in EE with excellent definitions and explanations.

FUNDAMENTALS

Alternating waveforms have currents and voltages that vary with time in a regular and symmetrical manner. Waveform shapes include square, sawtooth, and triangular, along with many variations on these themes. However, for most applications in electrical engineering the variations are sinusoidal in time. In this book, unless otherwise specified, currents and voltages are sinusoidal.[2]

When sinusoidal, the waveform is nearly always referred to as AC—that is, alternating current—indicating that the current is produced by the application of a sinusoidal voltage. This means that the flow of electrons changes directions, unlike DC circuits, where the flow of electrons is unidirectional (though the magnitude can change in time).

A circuit is said to be in a steady-state condition if the current and voltage time variation is purely constant (DC) or purely sinusoidal (AC).[3] In this book, unless otherwise specified, all circuits are in a steady-state condition.

All the methods, basic definitions, and equations involving DC circuits are applicable to AC circuits as well. AC electrical parameters have both magnitudes and angles. Nevertheless, following common practice, phasor notation will be used only if needed to avoid confusion.

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[2] Nearly all periodic functions can be represented as the sum of sinusoidal functions, which simplifies the mathematics needed.
[3] Steady-state AC may have a DC offset.

VOLTAGE

Sinusoidal variables can be expressed in terms of sines or cosines without any loss of generality.[4] A sine waveform is often the standard. If this is the case, Eq. 1 gives the value of the instantaneous voltage as a function of time.

Equation 1: Instantaneous Voltage

$$v(t) = V_m \sin(\omega t + \theta)$$

The maximum value of the sinusoid is given the symbol V_m and is also known as the *amplitude*. If $v(t)$ is not zero at $t = 0$, the sinusoid is *shifted* and a *phase angle*, θ, must be used, as shown in Fig.1.[5] Also shown in Fig. 1 is the period, T, which is the time that elapses in one cycle of the sinusoid. The *cycle* is the smallest portion of the sinusoid that repeats.

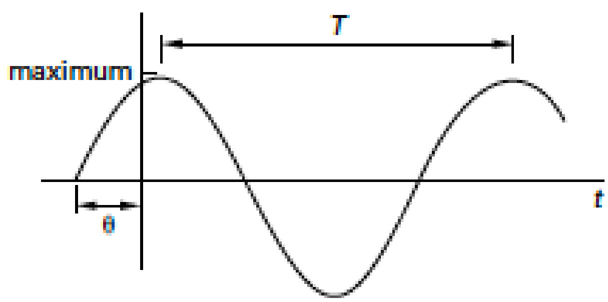


Figure 1: Sinusoidal Waveform with Phase Angle

Because the horizontal axis of the voltage in Fig. 1 corresponds to time, not distance, the waveform does not have a wavelength. The frequency, f, of the sinusoid is the reciprocal of the period in hertz (Hz). The angular frequency, ω, in rad/s can also be used.

Equation 2: Frequency

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

[4] The point at which time begins—that is, where t = 0—is of no consequence in steady-state AC circuit problems, as the signal repeats itself every cycle. Therefore, it makes no difference whether a sine or cosine waveform is used (though, if one exists, care must be taken to keep the phase angle correct).
[5] The term phase is not the same as phase difference, which is the difference between corresponding points on two sinusoids of the same frequency.

Equation 3: Angular Frequency

$$\omega = 2\pi f = \frac{2\pi}{\mathrm{T}}$$

An AC voltage waveform without a phase angle is plotted as a function of differing variables in Fig. 2.

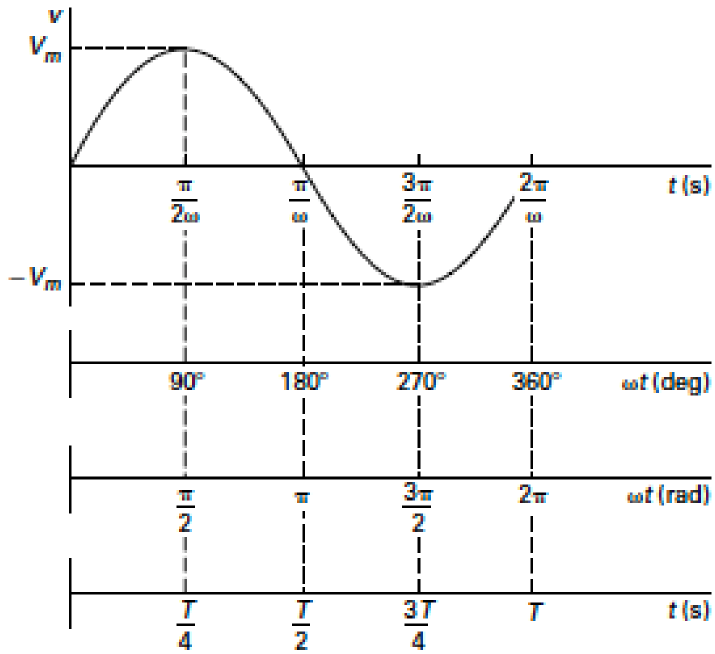


Figure 2: Sine Wave Plots

Exponentials, $e^{j\theta}$ and $e^{-j\theta}$, can be combined to produce sin θ and cos θ terms. As a result, sinusoids can be represented in the following equivalent forms.

- Trigonometric: $V_m \sin(\omega t + \theta)$
- Exponential: $V_m e^{j\theta}$
- Polar or Phasor: $V_m \angle \theta$
- Rectangular: $V_r + jV_i$

The use of complex exponentials and phasor analysis allows sinusoidal functions to be more easily manipulated mathematically, especially when dealing with derivatives. Exponents are manipulated algebraically, and the resulting sinusoid is then recovered using Euler's relation. This avoids complicated trigonometric mathematics. The angles used in exponentials and with angular frequency, ω, must be in radians. From Euler's relation,

Equation 4 Euler's Identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Equation 5: The Value of *j*

$$j = \sqrt{-1} = 1\angle\frac{\pi}{2}$$

Therefore, to regain the sinusoid from the exponential, use Euler's equation, keeping in mind two factors. First, either the real or the imaginary part contains the sinusoid desired, not both.[6] Second, the exponential should not be factored out of any equation until all the derivatives, or integrals, have been taken.

CURRENT

Current is the net transfer of electric charge per unit time. When a circuit's driving force, the voltage, is sinusoidal, the resulting current is also (though it may differ by an amount called the *phase angle difference*). The equations and representations shown for voltage, as well as any other representations of sinusoidal waveforms provided in this course, apply to current as well as to voltage.

IMPEDANCE

Electrical impedance, also known as *complex impedance*, is the total opposition a circuit presents to alternating current. It is equal to the ratio of the complex voltage to the complex current. Impedance, then, is a ratio of phasor quantities and is not itself a function of time. Relating voltage

[6] The cosine function could be used as the reference sinusoid and is in many texts. Over time, many standards have come to use the cosine as the preferred reference. In this text, the sine reference is used when graphing, since this allows the value at $t = 0$ to be zero. Otherwise, the cosine reference will be used. The desired form can be determined from the representation of the voltage (or current) source waveform as either a cosine or a sine function (see Eq. 1). Either way, only the phase angle changes.

and current in this manner is analogous to Ohm's law, which for AC analysis is referred to as *extended Ohm's law*. Impedance is given the symbol **Z** and is measured in ohms. The three passive circuit elements (resistors, capacitors, and inductors), when used in an AC circuit, are assigned an angle, ϕ, known as the *impedance angle*. This angle corresponds to the *phase difference angle* produced when a sinusoidal voltage is applied across that element alone.

Impedance is a complex quantity with a magnitude and associated angle. It can be written in *phasor form*—also known as *polar form*—for example, $\mathbf{Z}\angle\phi$, or in rectangular form as the complex sum of its resistive (R) and reactive (X) components.

Equation 6: Algebra of Complex Numbers

$$\mathbf{Z} \equiv R + jX$$

Equation 7: Real Part of Complex Number

$$R = Z\cos\phi \quad \text{[Resistive or Real Part]}$$

Equation 8: Reactive Part of Complex Number

$$X = Z\sin\phi \quad \text{[Reactive or Imaginary Part]}$$

The resistive and reactive components can be combined to form an *impedance triangle*. Such a triangle is shown in Fig. 3.

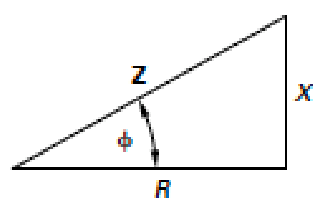


Figure 3: Impedance Triangle

The characteristics of the passive elements in AC circuits are given in Table 1.

Table 1: Characteristics of Resistors, Capacitors, and Inductors

	resistor	capacitor	inductor
value	R (Ω)	C (F)	L (H)
reactance, X	0	$\frac{-1}{\omega C}$	ωL
rectangular impedance, **Z**	$R + j0$	$0 - \frac{j}{\omega C}$	$0 + j\omega L$
phasor impedance, **Z**	$R\angle 0°$	$\frac{1}{\omega C}\angle -90°$	$\omega L\angle 90°$
phase	in-phase	leading	lagging
rectangular admittance, **Y**	$\frac{1}{R} + j0$	$0 + j\omega C$	$0 - \frac{j}{\omega L}$
phasor admittance, **Y**	$\frac{1}{R}\angle 0°$	$\omega C\angle 90°$	$\frac{1}{\omega L}\angle -90°$

ADMITTANCE

The reciprocal of impedance is called *admittance*, **Y**. Admittance is useful in analyzing parallel circuits, as admittances can be added directly. The defining equation is

Equation 9: Admittance

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{Z}\angle -\phi$$

The reciprocal of the resistive portion of an impedance is known as *conductance*, *G*. The reciprocal of the reactive part is termed the *susceptance*, *B*.

Equation 10: Conductance

$$G = \frac{1}{R}$$

Equation 11: Susceptance

$$B = \frac{1}{X}$$

Using these definitions and multiplying by a complex conjugate, admittance can be written in terms of resistance and reactance. Using the same method, impedance can be written in terms of conductance and susceptance. Equation 12 and Eq. 13 show this and can be used for conversion between admittance and impedance and vice versa.

Equation 12: Admittance in Terms of G and B

$$\mathbf{Y} = G + jB = \frac{R}{R^2 + X^2} - j\left(\frac{X}{R^2 + X^2}\right)$$

Equation 13: Impedance in Terms of G and B

$$\mathbf{Z} = R + jX = \frac{G}{G^2 + B^2} - j\left(\frac{B}{G^2 + B^2}\right)$$

VOLTAGE SOURCES

The energy for AC voltage sources comes primarily from electromagnetic induction. The concepts of ideal and real sources, as well as regulation, apply to AC sources. Independent sources deliver voltage and current at their rated values regardless of circuit parameters. Dependent sources, often termed *controlled sources*, deliver voltage and current at levels determined by a voltage or current somewhere else in the circuit. These types of sources occur in electronic circuitry and are also used to model electronic elements, such as transistors. The symbols used for AC voltage sources are shown in Fig. 4.

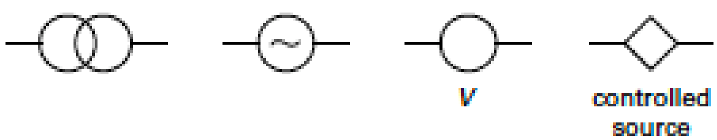


Figure 4: AC Voltage Source Symbols

AVERAGE VALUES

In purely mathematical terms, the average value of a periodic waveform is the first term of a Fourier series representing the function; that is, it is the zero frequency or DC value. If a function, *f(t)*, repeats itself in a time period *T*, then the average value of the function is given by Eq. 14. In Eq. 14, t_1 is any convenient time for evaluating the integral (i.e., the time that simplifies the integration). The integral itself is computed over the period.

Equation 14: Average Value

$$f_{ave} = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)\,dt$$

Integration can be interpreted as the area under the curve of the function. Equation 14 divides the net area of the waveform by the period *T*. This concept is illustrated in Fig. 5 and stated in mathematical terms by Eq. 15.

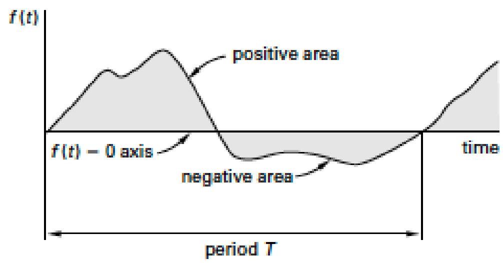


Figure 5: Average Value Areas Defined

Equation 15: Average Interpretation

$$f_{ave} = \frac{\text{positive area} - \text{negative area}}{T}$$

For the function shown in Fig. 5, the area above the axis is called the positive area and the area below is called the negative area. The average value is the net area remaining after the negative area is subtracted from the positive area and the result is divided by the period.

For any periodic voltage, Eq. 16 calculates the average value.

Equation 16: Average Voltage

$$V_{ave} = \frac{1}{2\pi}\int_{0}^{2\pi} v(\theta)\,d\theta$$
$$= \frac{1}{T}\int_{0}^{T} v(t)\,dt$$

Any waveform that is symmetrical with respect to the horizontal axis will result in a value of zero for Eq. 6. While mathematically correct, the electrical effects of such a voltage occur on both the positive and negative half-cycles. Therefore, the average is instead taken over only half a cycle. This is equivalent to determining the average of a *rectified* (i.e., the absolute value of the waveform). The average voltage for a rectified sinusoid is

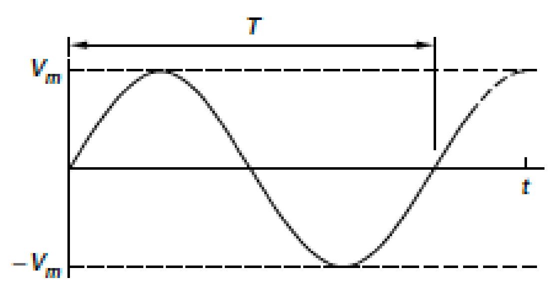
Equation 17: Average Value of Rectified Sinusoid

$$V_{ave} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\,d\theta$$
$$= \frac{2V_m}{\pi} \quad \text{[rectified sinusoid]}$$

A DC current equal in value to the average value of a rectified AC current produces the same electrolytic effects, such as capacitor charging, plating operations, and ion formation. Nevertheless, not all AC effects can be accounted for using the average value. For instance, in a typical DC meter, the average DC current determines the response of the needle. An AC current sinusoid of equal average magnitude will not result in the same effect since the torque on each half-cycle is in opposite directions, resulting in a net zero effect. Unless the AC signal is rectified, the reading will be zero.

Example 1

What is the average value of the pure sinusoid shown?



Solution

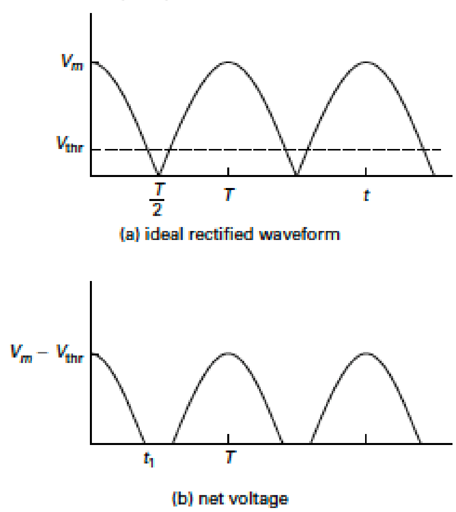
The sinusoid shown is a sine wave. Substituting into Eq. 14 with $t_1 = 0$ gives the following.

$$
\begin{aligned}
f_{ave} &= \frac{1}{T}\int_{t_1}^{t_1+T} f(t)\,dt = \frac{1}{T}\int_0^T V_m \sin t\,dt \\
&= \left(\frac{V_m}{T}\right)\left(-\cos t\right)\Big|_0^T = \left(\frac{V_m}{T}\right)\left(-\cos T + \cos 0\right) \\
&= \left(\frac{V_m}{T}\right)\left(-1+1\right) = 0
\end{aligned}
$$

The result is as expected, because a pure sinusoid has equal positive and negative areas.

Example 2

Diodes are used to rectify AC waveforms. Real diodes will not pass current until a threshold voltage is reached. As a result, the output is the difference between the sinusoid and the threshold value. This is illustrated in the following figures.



(a) ideal rectified waveform

(b) net voltage

What is the expression for the average voltage on the output of a real rectifier diode with a sinusoidal voltage input?

Solution

Because of the symmetry, the average value can be found using half the average determined from

Eq. 16 by integrating from zero to $T/2$, simplifying the calculation. The waveform is represented by the following.

$$v(t) = \begin{cases} V_m \cos \frac{\pi}{T} t - V_{thr} & [\text{for } 0 < t < t_1] \\ 0 \quad [\text{for } t_1 < t < T/2] \end{cases}$$

At $t = t_1$, the following condition exists.

$$V_m \cos \frac{\pi}{T} t_1 = V_{thr}$$

The equation for one-half the average is

$$\begin{aligned} \frac{1}{2} v_{ave} &= \frac{1}{T} \int_0^{t_1} \left(V_m \cos \frac{\pi}{T} t - V_{thr} \right) dt \\ &= \left(\frac{1}{T} \right) \left(\frac{T}{\pi} \right) V_m \sin \frac{\pi}{T} t_1 - V_{thr} \left(\frac{t_1}{T} \right) \end{aligned}$$

Using the equation for the condition at $t = t_1$ and the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$ results in the following equations.

$$\begin{aligned} \sin \frac{\pi}{T} t_1 &= \sqrt{1 - \left(\frac{V_{thr}}{V_m} \right)^2} \\ \frac{t_1}{T} &= \frac{1}{\pi} \arccos \frac{V_{thr}}{V_m} \end{aligned}$$

Substituting into the equation for one-half the average and rearranging gives the following final result.

$$v_{ave} = \frac{2}{\pi} V_m \sqrt{1 - \left(\frac{V_{thr}}{V_m} \right)^2} - \frac{2}{\pi} V_{thr} \arccos \frac{V_{thr}}{V_m}$$

The arccos must be expressed in radians. The conclusion drawn is that real diode average values are more complex than values for ideal diode full-wave rectification. However, if ideal diodes are assumed—that is, if the threshold voltage is considered negligible—the following errors are generated.

- 1% error for $V_{thr}/V_m = 0.0064$
- 2% error for $V_{thr}/V_m = 0.0128$
- 5% error for $V_{thr}/V_m = 0.0321$
- 10% error for $V_{thr}/V_m = 0.0650$

For most practical applications, the ideal diode assumption results in an error less than 10%.

ROOT-MEAN-SQUARE VALUE

In purely mathematical terms, the *effective*, or *root-mean-square* (rms) *value*, of a periodic waveform represented by a function, $f(t)$, which repeats itself in a time period, is given by Eq. 18. In Eq. 18, t_1 is any convenient time for evaluating the integral (i.e., the time that simplifies the integration). The integral itself is computed over the period.

Equation 18: RMS/Effective Integral

$$f_{\text{rms}}^2 = \frac{1}{T}\int_{t_1}^{t_1+T} f^2(t)\,dt$$

For any periodic voltage, Eq. 19 calculates the effective or rms value.

Equation 19: RMS/Effective Voltage

$$\begin{aligned} V &= V_{eff} = V_{rms} \\ &= \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt} \\ &= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(\theta)\,d\theta} \end{aligned}$$

Normally when a voltage or current variable, V or I, is left unsubscripted, it represents the effective (rms) value.[7] Rectification of the waveform is not necessary to calculate the effective value. (The squaring of the waveform ensures the net result of integration is something other than zero for a sinusoid.) For a sinusoidal waveform, $V = V_m/\sqrt{2} \approx 0.707V_m$. A DC current of magnitude I produces the same heating effect as an AC current of magnitude I_{eff}.

Table 2 gives the characteristics of various commonly encountered alternating waveforms. In this table, two additional terms are introduced. The *form factor* is given by

Equation 20: Form Factor

$$\text{FF} = \frac{V_{eff}}{V_{ave}}$$

The *crest factor*, CF, also known as the *peak factor* or *amplitude factor*, is

Equation 21: Crest Factor

$$\text{CF} = \frac{V_m}{V_{eff}}$$

Nearly all periodic functions can be represented by a Fourier series. A Fourier series can be written as follows.

[7] The value of the standard voltage in the United States, reported as 115–120 V, is an effective value.

Equation 22: Fourier Series

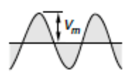
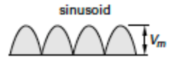
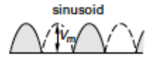
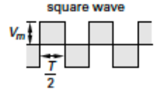
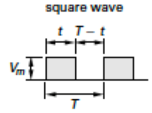
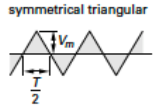
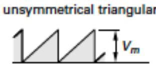
$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2\pi n}{T}t+\sum_{n=1}^{\infty}b_n\sin\frac{2\pi n}{T}t$$

The average value is the first term. The rms value is as follows.

Equation 23: RMS Value

$$f_{rms}=\sqrt{\left(\frac{1}{2}a_0\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)}$$

Table 2: Characteristics of Alternating Waveforms

waveform	$\frac{V_{ave}}{V_m}$	$\frac{V_{rms}}{V_m}$	FF	CF
sinusoid 	0	$\frac{1}{\sqrt{2}}$	–	$\sqrt{2}$
full-wave rectified sinusoid 	$\frac{2}{\pi}$	$\frac{1}{\sqrt{2}}$	$\frac{\pi}{2\sqrt{2}}$	$\sqrt{2}$
half-wave rectified sinusoid 	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{\pi}{2}$	2
symmetrical square wave 	0	1	–	1
unsymmetrical square wave 	$\frac{t}{T}$	$\sqrt{\frac{t}{T}}$	$\sqrt{\frac{T}{t}}$	$\sqrt{\frac{T}{t}}$
sawtooth and symmetrical triangular 	0	$\frac{1}{\sqrt{3}}$	–	$\sqrt{3}$
sawtooth and unsymmetrical triangular 	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$

Example 3

A peak sinusoidal voltage, V_p, of 170 V is connected across a 240 Ω resistor in a lightbulb. What is the power dissipated by the bulb?

Solution

From Table 2, the effective voltage is as follows.

$$V = \frac{V_p}{\sqrt{2}} = \frac{170 \text{ V}}{\sqrt{2}}$$
$$= 120.21 \text{ V}$$

The power dissipated is as follows.

$$P = \frac{V^2}{R}$$
$$= \frac{(120.21 \text{ V})^2}{240 \ \Omega}$$
$$= 60.21 \text{ W}$$

Example 4

What is the rms value of a constant 3 V signal?

Solution

Using Eq. 18, the rms value is as follows.

$$v_{rms}^2 = \frac{1}{T}\int_0^T (3 \text{ V})^2 \, dt$$
$$= \left(\frac{1}{T}\right)(9 \text{ V}^2)(T-0) = 9 \text{ V}^2$$
$$v_{rms} = \sqrt{9 \text{ V}^2}$$
$$= 3 \text{ V}$$

PHASE ANGLES

AC circuit inductors and capacitors have the ability to store energy in magnetic and electric fields, respectively. Consequently, while the same shape, the voltage and current waveforms differ by an amount called the *phase angle difference*, ϕ. Ordinarily, the voltage and current sinusoids do not peak at the same time. In a *leading circuit*, the phase angle difference is positive and the current peaks before the voltage (see Fig. 6). A leading circuit is termed a *capacitive circuit*. In a *lagging circuit*, the phase angle difference is negative and the current peaks after the voltage. A lagging circuit is termed an *inductive circuit*. These cases are represented mathematically as follows.

Equation 24: Voltage Reference

$$v(t) = V_m \sin\left(\omega t + \theta\right) \quad \text{[reference]}$$

Equation 25: Leading Current

$$i(t) = I_m \sin(\omega t + \theta + \phi) \quad \text{[leading]}$$

Equation 26: Lagging Current

$$i(t) = I_m \sin(\omega t + \theta - \phi) \quad \text{[lagging]}$$

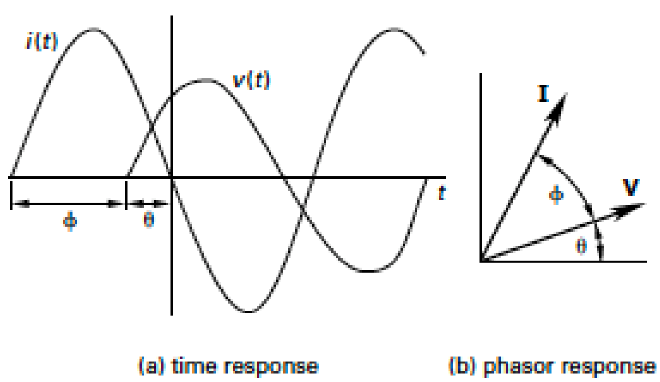


Figure 6: Leading Phase Angle Difference

SINUSOID

The most important waveform in electrical engineering is the sine function, or a waveform that is sinusoidal in value with respect to time. A sinusoidal waveform has an amplitude (also known as a *magnitude* or *peak value*) that remains constant, such as V_m. The waveform, however, repeats or goes through cycles. From Eq. 1, the sinusoid can be characterized by three quantities: magnitude, frequency, and phase angle.[8] The major properties of the sinusoid were covered earlier.

A circuit processes or changes waveforms. This is called *signal processing* or *waveform processing*. *Signal analysis* is the determination of these waveforms. The sinusoid can be represented as a phasor and used to analyze electrical circuits only if the circuit is in a steady-state condition. The terms AC and DC imply steady-state conditions. For AC circuits, *sinusoidal steady-state* indicates that all voltages and currents within the circuit are sinusoids of the same frequency as the excitation— that is, the driving voltage.

Example 5

A reference voltage of $v(t) = 170 \sin \omega t$ is used during an experiment. A measurement of a second voltage, $v_2(t)$ occurs. The second voltage reaches its peak 2.5 ms before the reference voltage and has the same peak value 20 ms later. The peak value is 1.8 times the reference peak. What is the expression for $v_2(t)$?

Solution

Any sinusoidal voltage can be represented as in Eq. 1.

$$v(t) = V_m \sin(\omega t + \theta)$$

The peak value (equivalent to V_m) is 1.8 times the reference, or 1.8 × 170 V = 306 V. Calculate the angular frequency using Eq. 3 and the given period of 20 ms.

---

[8] A phase angle of ±90° changes the sine function into a cosine function. Also, frequency refers in this case to the angular frequency, ω, in radians per second. The term frequency also refers to the term *f*, measured in hertz (Hz).

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\ &= \frac{2\pi}{20\ \text{ms}} \\ &= 314\ \text{rad/s}\end{aligned}$$

The phase angle is determined by changing the time, 2.5 ms, into its corresponding angle, θ, as follows.

$$\begin{aligned}\theta &= \omega t \\ &= \left(314\ \frac{\text{rad}}{\text{s}}\right)\left(2.5\times 10^{-3}\ \text{s}\right) \\ &= 0.79\ \text{rad}\end{aligned}$$

Calculating 0.79 rad give a result 45°. Substituting the calculated values gives the result.

$$\begin{aligned}v_2(t) &= V_m \sin\left(\omega t + \theta\right) \\ &= 306 \sin\left(314t + 45^\circ\right)\end{aligned}$$

The result mixes the use of radians (314) and degrees (45). This is often done for clarity. Ensure one or the other is converted prior to calculations. It is often best to deal with all angles in radians.

PHASORS

The addition of voltages and currents of the same frequency is simplified mathematically by treating them as phasors.[9] Using a method called the *Steinmetz algorithm*, the following two quantities, introduced earlier, are considered analogs.

Equation 27: Trigonometric & Phasor Representations

$$v(t) = V_m \sin(\omega t + \theta) \text{ and } V = V_m \angle\theta$$

[9] For all circuit elements at the same frequency, the circuit must be in a steady-state condition.

Both forms show the magnitude and the phase, but the phasor, $V_m \angle\theta$, does not show frequency. In the phasor form, the frequency is implied. The phasor form of Eq. 27 is shown in Fig. 7.

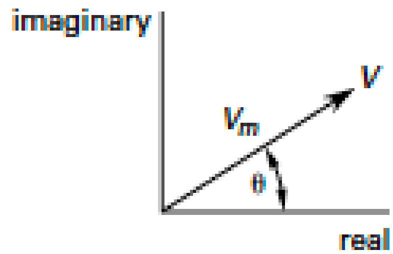


Figure 7: Magnitude and Angle of a Phasor

The phasor is actually a point, but is represented by an arrow of magnitude V_m at an angle θ with respect to a reference, normally taken to be $\theta = 0$. A reference phasor is one of known value, usually the driving voltage of a circuit (i.e., the voltage phasor). The reference phasor would be shown in the position of the real axis with θ equal to zero. Phasors are summed using phasor addition, commonly referred to as *vector addition*.[10] The Steinmetz algorithm is illustrated in Fig. 8.

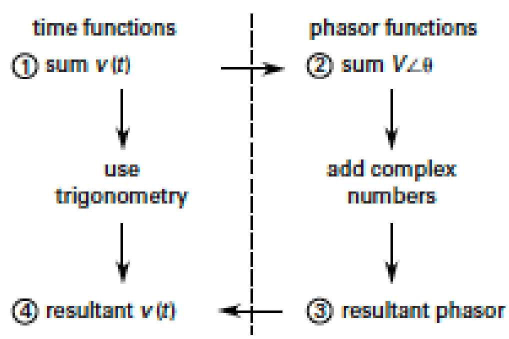


Figure 8: Steinmetz Algorithm Steps

In electric circuits with sinusoidal waveforms, an alternate phasor representation is often used. The alternate representation is called the *effective value phasor notation*. In this notation, the rms values

[10] The term "vector addition" is not technically correct because phasors, with the exception of impedance and admittance phasors, rotate with time. The methods, however, are identical.

of the voltage and current are used instead of the peak values. The angles are also given in degrees. The phasor notation of Eq. 27 is then modified to that of Eq. 28.

Equation 28: RMS Phasor Representation

$$\begin{aligned} V &= V_{rms}\angle\theta \\ &= \frac{V_m}{\sqrt{2}}\angle\theta \\ &= V\angle\theta \end{aligned}$$

The effective value phasor notation represented by the first and second parts of Eq. 28 is used by NCEES. When the notation is not specified (for example, when a subscript is not used as in the third part of Eq. 28), assume the effective notation. Additionally, many texts do not show phasors in bold font, but instead let the angle symbol indicate that a phasor is being used. Care should be taken to determine the type of notation in use.

Ensure that angles substituted into exponential forms are in radians or errors in mathematical calculations will result.

COMPLEX REPRESENTATION

Phasors are plotted in the complex plane. Figure 9 shows the relationships among the complex quantities introduced.

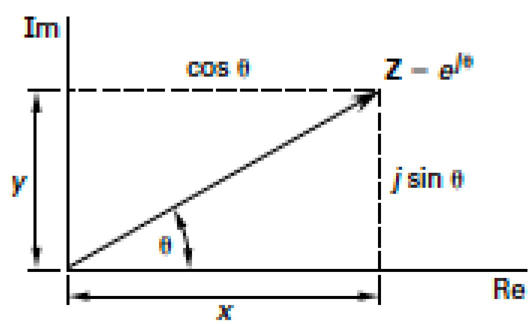


Figure 9: Complex Quantities

In Fig. 9, **Z** represents a complex number, not the impedance. Equation 1 is repeated here for convenience.

Equation 29: Instantaneous Voltage Trigonometric Form

$$v(t) = V_m \sin(\omega t + \theta)$$

Using Euler's relation, Eq. 4, the voltage *v*(*t*) can be represented as

Equation 30: Voltage Exponential Form

$$v(t) = V_m e^{j(\omega t + \theta)}$$

Consequently, *v*(*t*) is the imaginary part of Eq. 30. If the cosine function is used, *v*(*t*) is the real part of Eq. 30. The fixed portion of Eq. 17.30 ($e^{j\theta}$) can be separated from the time-variable portion ($e^{j\omega t}$) of the function.

This is shown in Fig. 10(a) and (b). The magnitude of $e^{j\omega t}$remains equal to one, but the angle increases (rotates counterclockwise) linearly with time. All functions with an $e^{j\omega t}$term are assumed to rotate counterclockwise with an angular velocity of ω in the complex plane.

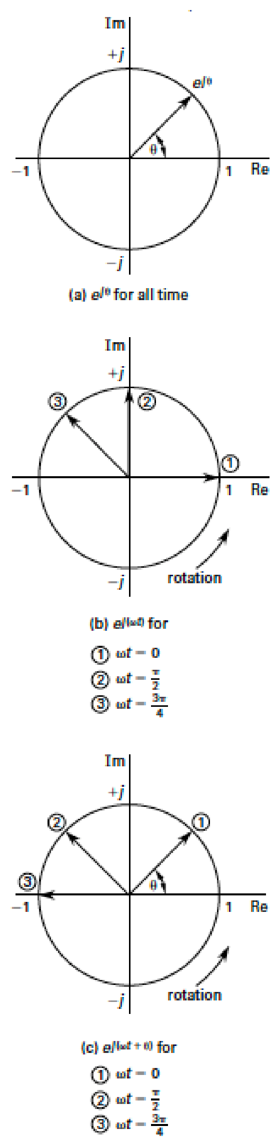


Figure 10: Phasor Rotation in the Complex Plane[11]

[11] The convention of measuring angles in a counterclockwise (positive) direction and clockwise (negative) direction is based on historical mathematical and geometric traditions. It is a convention that has been widely adopted and used consistently in mathematics, physics, and engineering. One reason for this convention is that it aligns with the direction of rotation of the unit circle in the Cartesian coordinate system. In this system, the counterclockwise direction is considered positive, while the clockwise direction is considered negative. This convention helps maintain consistency and simplifies calculations in various mathematical and scientific contexts. [*Source: Quora*]

The voltage can be represented as in Fig. 10(c) and

Equation 31: Voltage Rotating & Exponential

$$v(t) = V_m e^{j\omega t} e^{j\theta}$$

In the rotating portion of Eq. 31 is assumed to exist, the voltage can be written as follows.

Equation 32: Voltage Exponential

$$v(\theta) = V_m e^{j\theta}$$

Changing Eq. 32 to the phasor form yields the following.

Equation 33: Voltage Phasor Form

$$\begin{aligned} \mathbf{V} &= V_m \angle\theta \quad \text{[maximum form]} \\ \mathbf{V} &= V_{rms} \angle\theta \quad \text{[rms form]} \\ &= \frac{V_m}{\sqrt{2}} \angle\theta \\ &= V \angle\theta \end{aligned}$$

The properties of complex numbers, which can represent voltage, current, or impedance, are summarized in Table 3. The designations used in the table are illustrated in Fig. 9.

Table 3: Properties of Complex Numbers

	rectangular form	polar/exponential form
	$\mathbf{Z} = x + jy$	$\mathbf{Z} = \lvert \mathbf{Z} \rvert \angle \theta$ $\mathbf{Z} = \lvert \mathbf{Z} \rvert e^{j\theta} = \lvert \mathbf{Z} \rvert \cos\theta + j \lvert \mathbf{Z} \rvert \sin\theta$
relationship between forms	$x = \lvert \mathbf{Z} \rvert \cos\theta$ $y = \lvert \mathbf{Z} \rvert \sin\theta$	$\lvert \mathbf{Z} \rvert = \sqrt{x^2 + y^2}$ $\theta = \arctan \frac{y}{x}$
complex conjugate	$\mathbf{Z}^* = x - jy$ $\mathbf{Z}\mathbf{Z}^* = (x^2 + y^2) = \lvert z \rvert^2$	$\mathbf{Z}^* = \lvert \mathbf{Z} \rvert e^{-j\theta} = \lvert \mathbf{Z} \rvert \angle -\theta$ $\mathbf{Z}\mathbf{Z}^* = (\lvert \mathbf{Z} \rvert e^{j\theta})(\lvert \mathbf{Z} \rvert e^{-j\theta}) = \lvert \mathbf{Z} \rvert^2$
addition	$\mathbf{Z}_1 + \mathbf{Z}_2 = (x_1 + x_2) + j(y_1 + y_2)$	$\mathbf{Z}_1 + \mathbf{Z}_2 = (\lvert \mathbf{Z}_1 \rvert \cos\theta_1 + \lvert \mathbf{Z}_2 \rvert \cos\theta_2)$ $+ j(\lvert \mathbf{Z}_1 \rvert \sin\theta_1 + \lvert \mathbf{Z}_2 \rvert \sin\theta_2)$
multiplication	$\mathbf{Z}_1\mathbf{Z}_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$	$\mathbf{Z}_1\mathbf{Z}_2 = \lvert \mathbf{Z}_1 \rvert \lvert \mathbf{Z}_2 \rvert \angle \theta_1 + \theta_2$
division	$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{\lvert \mathbf{Z}_2 \rvert^2}$	$\frac{z_1}{z_2} = \frac{\lvert \mathbf{Z}_1 \rvert}{\lvert \mathbf{Z}_2 \rvert} \angle \theta_1 - \theta_2$

RESISTORS

Resistors oppose the movement of electrons. In an ideal or pure resistor, no inductance or capacitance exists. The magnitude of the impedance is the resistance, *R*, with units of ohms and an impedance angle of zero. Therefore, voltage and current are in phase in a purely resistive circuit.

Equation 34: Resistance

$$\mathbf{Z}_R = R\angle 0^\circ = R + j0$$

CAPACITORS

Capacitors oppose the movement of electrons by storing energy in an electric field and using this energy to *resist changes in voltage* over time. Unlike a resistor, an ideal or perfect capacitor consumes no energy. Equation 35 gives the impedance of an ideal capacitor with capacitance *C*. The magnitude of the impedance is termed the capacitive reactance, X_C, with units of ohms and an impedance angle of −π/2 (−90°). Consequently, the current leads the voltage by 90° in a purely capacitive circuit.[12]

[12] The impedance angle for a capacitor is negative, but the current phase angle difference is positive—hence the term "leading." This occurs mathematically because the current is obtained by dividing the voltage by the impedance: **I** = **V**/**Z**.

Equation 35: Impedance of Capacitors

$$\mathbf{Z}_C = X_C \angle -90^\circ = 0 + jX_C$$

Equation 36: Capacitive Reactance

$$X_c = \frac{-1}{\omega C} = \frac{-1}{2\pi f C}$$

INDUCTORS

Inductors oppose the movement of electrons by storing energy in a magnetic field and using this energy to *resist changes in current* over time. Unlike a resistor, an *ideal* or *perfect inductor* consumes no energy. Equation 37 gives the impedance of an ideal inductor with inductance *L*. The magnitude of the impedance is termed the inductive reactance, X_L, with units of ohms and an impedance angle of π/2 (90°). Consequently, the current lags the voltage by 90° in a purely inductive circuit.[13]

Equation 37: Impedance of Inductors

$$\mathbf{Z}_L = X_L \angle 90^\circ = 0 + jX_L$$

Equation 38: Inductive Reactance

$$X_L = \omega L = 2\pi f L$$

COMBINING IMPEDANCES

Impedances in combination are like resistors: Impedances in series are added, while the reciprocals of impedances in parallel are added. For series circuits, the resistive and reactive parts of each impedance element are calculated separately and summed. For parallel circuits, the conductance and susceptance of each element are summed. The total impedance is found by a complex addition

[13] The impedance angle for an inductor is positive, but the current phase angle difference is negative—hence the term "lagging." This occurs mathematically because the current is obtained by dividing the voltage by the impedance: **I** = **V**/**Z**.

of the resistive (conductive) and reactive (susceptive) parts. It is convenient to perform the addition in rectangular form.

Equation 39: Impedance in Series

$$Z_e = \sum Z = \sqrt{\left(\sum R\right)^2 + \left(\sum X_L - \sum X_C\right)^2} \quad \text{[series]}$$

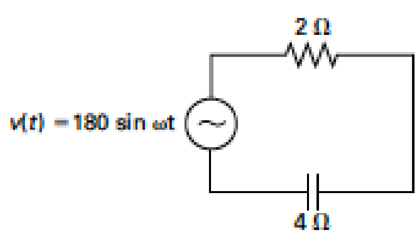
Equation 40: Impedance in Parallel

$$\frac{1}{Z_e} = \sum \frac{1}{Z} = Y_e = \sqrt{\left(\sum G\right)^2 + \left(\sum B_L - \sum B_C\right)^2} \quad \text{[parallel]}$$

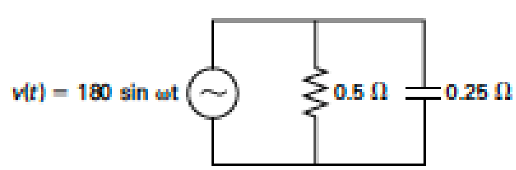
Example 6

Determine the impedance and admittance of the following circuits.

(a)



(b)



Solution

(a) From Eq. 39,

$$Z = \sqrt{R^2 + Z_C^2} = \sqrt{(2\ \Omega)^2 + (4\ \Omega)^2} = 4.47\ \Omega$$

$$\phi = \arctan \frac{X_C}{R} = \arctan \frac{-4\ \Omega}{2\ \Omega} = -63.4^\circ$$

$$\mathbf{Z} = 4.47\ \Omega\angle -63.4^\circ$$

From Eq. 9,

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{4.47\ \Omega\angle -63.4^\circ} = 0.224\ \mathrm{S}\angle 63.4^\circ$$

(b) Because this is a parallel circuit, work with the admittances.

$$G = \frac{1}{R} = \frac{1}{0.5\ \Omega} = 2\ \mathrm{S}$$

$$B_C = \frac{1}{X_C} = \frac{1}{0.25\ \Omega} = 4\ \mathrm{S}$$

$$Y = \sqrt{G^2 + B_C^2} = \sqrt{(2\ \mathrm{S})^2 + (4\ \mathrm{S})^2} = 4.47\ \mathrm{S}$$

$$\phi = \arctan \frac{B_C}{G} = \arctan \frac{4\ \mathrm{S}}{2\ \mathrm{S}} = 63.4^\circ$$

$$\mathbf{Y} = 4.47\ \mathrm{S}\angle 63.4^\circ$$

$$\mathbf{Z} = \frac{1}{\mathbf{Y}} = \frac{1}{4.47\ \mathrm{S}\angle 63.4^\circ} = 0.224\ \Omega\angle -63.4^\circ$$

OHM'S LAW

Ohm's law for AC circuits with linear circuit elements is similar to Ohm's law for DC circuits.[14] The difference is that all the terms are represented as phasors.[15]

Equation 41: Ohm's Law for AC Circuits

$$\mathbf{V} = \mathbf{IZ}$$

Equation 42: Ohm's Law Magnitude and Phase

$$V\angle\theta_v = \left(I\angle\phi_I\right)\left(Z\angle\phi_Z\right)$$

POWER

The instantaneous power, *p*, in a purely resistive circuit is

Equation 43: Instantaneous Power

$$\begin{aligned} p_R(t) &= i(t)v(t) \\ &= \left(I_m \sin\omega t\right)\left(V_m \sin\omega t\right) \\ &= I_m V_m \sin^2 \omega t \\ &= \tfrac{1}{2} I_m V_m - \tfrac{1}{2} I_m V_m \cos 2\omega t \end{aligned}$$

The second term of Eq. 43 integrates to zero over the period. Therefore, the average power dissipated is as follows. Note in the equation the transition from maximum (peak) value to rms in the second row of the equations. Equation 28 was used to change the maximum, or peak, values into the more usable rms, or effective, values. Further, the "rms" subscript is usually not shown.

[14] Ohm's law can be used on *nonlinear devices* (NLD) if the region analyzed is restricted to be approximately linear. When this condition is applied, the analysis is termed *small-signal analysis*.

[15] In general, phasors are considered to rotate with time. A secondary definition of a phasor is any quantity that is a complex number. As a result, the impedance is also considered a phasor even though it does not change with time.

Equation 44: Complex Power

$$
\begin{aligned}
P_R &= \tfrac{1}{2} I_m V_m = \frac{V_m^2}{2R} \\
&= \left(\frac{I_m}{\sqrt{2}}\right)\left(\frac{V_m}{\sqrt{2}}\right) \\
&= I_{rms} V_{rms} = IV
\end{aligned}
$$

In a purely capacitive circuit, the current leads the voltage by 90°.[16] This allows the current term of Eq. 43 to be replaced by a cosine (because the sine and cosine differ by a 90° phase). The instantaneous power in a purely capacitive circuit is

Equation 45: Instantaneous Power—Capacitor[17]

$$
\begin{aligned}
p_C(t) &= i(t)v(t) \\
&= \left(I_m \cos \omega t\right)\left(V_m \sin \omega t\right) \\
&= I_m V_m \sin \omega t \cos \omega t \\
&= \tfrac{1}{2} I_m V_m \sin 2\omega t
\end{aligned}
$$

Because the sin $2\omega t$ term integrates to zero over the period, the average power is zero. Nevertheless, power is stored during the charging process in the *electric field* and returned to the circuit during the discharging process.

Equation 46: Average Power—Capacitor

$$P_C = 0$$

---

16 A memory aid is ELI the ICE man. The L is for an inductor and the C is for the capacitor. Then reading in order, ELI becomes "Voltage (E) leads Current (I)" in an Inductive circuit. "Current (I) leads the Voltage (E)" in a Capacitive circuit.

17 Instantaneous Power can be either positive or negative. A *positive instantaneous power flow* indicates that power flows from *source to the load*, while a *negative instantaneous power flow* indicates that power flows from *load to source*.

Similarly, the instantaneous power in an inductor is

Equation 47: Instantaneous Power—Inductor[18]

$$p_L(t) = -\tfrac{1}{2} I_m V_m \sin 2\omega t$$

Again, the sin $2\omega t$ integrates to zero over the period, and the average power is zero. Nevertheless, power is stored during the expansion of the *magnetic field* and returned to the circuit during the contraction of the magnetic field.

Equation 48: Average Power—Inductor

$$P_L = 0$$

REAL POWER AND THE POWER FACTOR

In a circuit that contains all three circuit elements (resistors, capacitors, and inductors) or the effects of all three, the average power is calculated from Eq. 14 as

Equation 49: Single Phase Average Power

$$P_{ave} = \frac{1}{T}\int_0^T i(t)v(t)\,dt$$

Let the generic voltage and current waveforms be represented by Eq. 50 and Eq. 51. The *current angle* ϕ is equal to $\theta \pm \phi$.[19]

Equation 50: Generic Current

$$i(t) = I_m \sin(\omega t + \phi)$$

[18] In the positive half of the voltage waveform between the angle of 0° and 90°, the inductor current is negative while the supply voltage is positive. Hence, the negative sign.
[19] Using a current angle simplifies the derivational mathematics (not shown) and clarifies the definition of the power factor angle given in this section

Equation 51: Generic Voltage

$$v(t) = V_m \sin(\omega t + \theta)$$

Substituting Eq. 50 and Eq. 51 into Eq. 49 gives the following.

Equation 52: Average Single Phase Power

$$P_{ave} = \left(\frac{I_m V_m}{2}\right)\cos\phi_{pf} = I_{rms} V_{rms} \cos\phi_{pf}$$

The angle $\phi_{pf} = |\theta - \phi|$ is the *power factor angle*. It represents the difference between the voltage and current angles. This difference is the impedance angle, $\pm\phi$. Because $\cos(-\phi) = \cos(+\phi)$, only the absolute value of the impedance angle is used. Because the absolute value is used in the equation, the terms *leading* (for a capacitive circuit) and *lagging* (for an inductive circuit) must be used when describing the power factor. The power factor of a purely resistive circuit equals one; the power factor of a purely reactive circuit equals zero.

Equation 52 determines the magnitude of the product of the current and voltage in phase with one another—that is, of a resistive nature. Consequently, Eq. 52 determines the power consumed by the resistive elements of a circuit. This average power is called the *real power* or *true power*, and sometimes the *active power*. The quantity $\cos\phi_{pf}$, or $\cos|\phi|$, is called the *power factor* or *phase factor* and given the symbol pf. The power factor is also *equal to the ratio of the real power to the apparent power* (see the Complex Power topic). Often, no subscript is used on P when it represents the real power, nor are subscripts used on rms or effective values. The real power is then given by

Equation 53: Real Power

$$P = IV\phi_{pf} = IVpf$$

It is important to realize that the real power cannot be obtained by multiplying phasors. Doing so results in the addition of the voltage and current angles when the difference is required. The incorrect result would be as follows.

Equation 54: Power Angle—Incorrect Result

$$P \neq \left(I\angle\phi_I\right)\left(V\angle\theta\right) = IV\angle\theta_Z + \phi_I$$

If phasor power is used, the difficulty can be alleviated by using Eq. 55—that is, using the complex conjugate for the current.

Equation 55: Power Angle—Correct Result

$$P = \mathrm{Re}\left(\mathbf{VI}^*\right)$$

Equation 55 is equivalent to Eq. 53. (Recall the conjugate makes the associated angle negative.)

REACTIVE POWER

The *reactive power*, Q, measured in units of VAR, is given by

Equation 56: Reactive Power

$$Q = IV\sin\phi_{pf}$$

The reactive power, sometimes called the *wattless power*, is the product of the rms values of the current and voltage multiplied by the *quadrature* of the current. The quantity sin ϕ_{pf} is called the *reactive factor*.[20] The reactive power represents the energy stored in the inductive and capacitive elements of a circuit.

APPARENT POWER

The apparent power, S, measured in voltamperes, is given by

Equation 57: Complex Power

$$S = IV$$

[20] The power factor angle, ϕ_{pf}, is used instead of the absolute value of the impedance angle (as in the power factor) because the sine function results in positive and negative values depending on the difference between the voltage and current angles. If ϕ_{pf} were not used, $\pm\phi$ would have to be used.

The apparent power is the product of the rms values of the voltage and current without regard to the angular relationship between them. The apparent power is representative of the combination of real and reactive power. As such, not all of the apparent power is dissipated or consumed. Nevertheless, electrical engineers must design systems to adequately handle this power as it exists in the system.

COMPLEX POWER AND THE POWER TRIANGLE

The real, reactive, and apparent powers can be related to one another as vectors. The *complex power vector,* **S**, is the vector sum of the *reactive power vector*, **Q**, and the **real power vector**, **P**. The *magnitude* of **S** is given by Eq. 57. The *magnitude* of **Q** is given by Eq. 56. The *magnitude* of **P** is given by Eq. 53. In general, **S** = **I*V**, where **I*** is the complex conjugate of the current—that is, the current with the phase difference angle reversed. This convention is arbitrary, but emphasizes that the *power angle*, ϕ, associated with **S** is the same as the overall impedance angle, ϕ, whose *magnitude* equals the power factor angle, ϕ_{pf}.

The relationship between the magnitudes of these powers is as follows.

Equation 58: Power Triangle Magnitudes

$$S^2 = P^2 + Q^2$$

A drawing of the power vectors in a complex plane is shown in Fig. 11 for leading and lagging conditions, and the following relationships are determined from it.

Equation 59: Real Power Magnitude

$$P = S\cos\phi$$

Equation 60: Reactive Power Magnitude

$$Q = \mathrm{S}\sin\phi$$

The ratio of reactance, *X*, to resistance, *R*, given in Eq. 62 and often used in fault analysis and transformer evaluation, can be determined from the power factor, pf, and vice versa. The power factor is the relationship between the current and the voltage. Since this relationship shifts significantly during a short circuit, in fault analysis the short-circuit power factor must be used.

Equation 61: Power Factor

$$pf = \frac{P}{S}$$

Equation 62: X/R Ratio

$$\frac{X}{R} = \tan\left(\arccos pf\right)$$

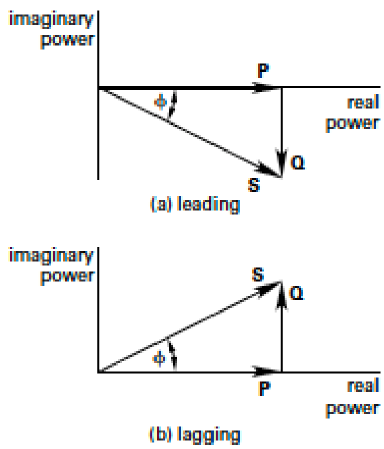
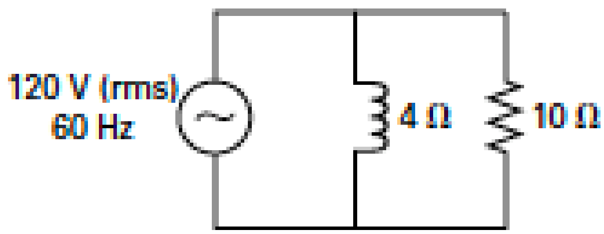


Figure 11: Power Triangle[21]

[21] Note how a lagging circuit uses +Q. Though this is embedded in the math, consider another memory aide: most circuits are lagging (and were noted first) and naturally given the positive value.

Example 7

For the circuit shown, find the (a) apparent power, (b) real power, and (c) reactive power; and (d) draw the power triangle.



Solution

The equivalent impedance for this parallel circuit is as follows.

$$\frac{1}{Z} = \frac{1}{j4\ \Omega} + \frac{1}{10\ \Omega} = -j0.25 + 0.10\ \text{S}$$

$$Z = 3.714\ \Omega$$

$$\phi_Z = \arctan\frac{0.25\ \text{S}}{0.10\ \text{S}} = 68.2^\circ$$

The total current is as follows.

$$I = \frac{V}{Z} = \frac{120\ \text{V}\angle 0^\circ}{3.714\ \Omega\angle 68.2^\circ} = 32.31\ \text{A}\angle -68.2^\circ$$

(a) The apparent power is as follows.

$$S = I^*V = (32.31\ \text{A}\angle 68.2^\circ)(120\ \text{V}\angle 0^\circ) = 3877\ \text{VA}\angle 68.2^\circ$$

(The angle of apparent power is usually not reported.)

(b) The real power is

$$P = \frac{V_R^2}{R} = \frac{(120\ \text{V})^2}{10\ \Omega} = 1440\ \text{W}$$

Alternatively, the real power can be calculated from Eq. 59.

$$P = S\cos\phi = (3877\ \text{VA})\cos 68.2^\circ = 1440\ \text{W}$$

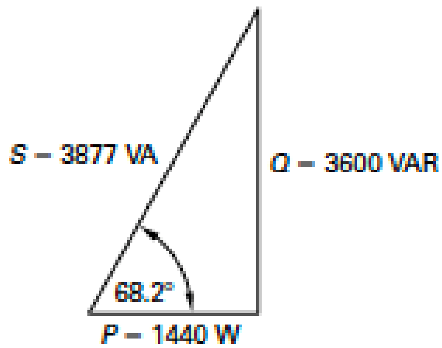
(c) The reactive power is

$$Q = \frac{V_L^2}{X_L} = \frac{(120\ \text{V})^2}{4\ \Omega} = 3600\ \text{VAR}$$

Alternatively, the reactive power can be calculated from Eq. 60.

$$Q = S\sin\phi = (3877\ \text{VA})\sin 68.2^\circ = 3600\ \text{VAR}$$

(d) The real power is represented by the vector (in rectangular form) $1440 + j\,0$. The reactive power is represented by the vector $0 + j\,3600$. The apparent power is represented (in phasor form) by the vector $3877\angle 68.2°$. The power triangle is



MAXIMUM POWER TRANSFER

Assuming a fixed primary impedance, the maximum power condition in an AC circuit is similar to that in a DC circuit. That is, the maximum power is transferred from the source to the load when the impedances match. This occurs when the circuit is in resonance. Resonance is discussed more fully in a future course. The conditions for maximum power transfer, and therefore resonance, are given by

Equation 63: Resistance for Maximum Power Transfer

$$R_{\text{load}} = R_s$$

Equation 64: Reactance for Maximum Power Transfer

$$X_{\text{load}} = -X_s$$

AC CIRCUIT ANALYSIS

All of the methods and equations presented in Chap. 16, such as Ohm's and Kirchhoff's laws and loop-current and node-voltage methods, can be used to analyze AC circuits as long as complex arithmetic is utilized.

SIGNAL TYPES

A number of different signal types are encountered in AC analysis. Each has distinctive uses and is best suited to distinctive analysis techniques. *Discrete signals* are used to sample analog values and convert these values into a form that can be used by electronic control circuitry. *Digital signals* are on/off or one/zero signals often used in computers and digital systems that monitor and manipulate control circuitry. Each signal type is shown in Fig. 12, and each part of the figure shows a different way of representing the same signal.

An analog signal represents a variable as a continuous measured parameter, as shown in Fig. 12(a).[22] A discrete signal is composed of separate and distinct parts. It represents a continuous time-varying signal at discrete time intervals known as the *sampling rate*, as shown in Fig. 12(b). The amplitude of the discrete signal is determined by the continuous signal that is sampled. A digital signal makes both the sampling rate (time) and the amplitude discrete, as shown in Fig. 17.12(c).

[22] An analog signal can also be defined as a signal that represents a time-varying feature (variable) of another time-varying quantity. A typical example is an AC sinusoidal voltage, which can represent a transducer output, an audio signal, or any of a multitude of sensor signals. In the case of an AC machine, the voltage does not represent another variable per se, though it may be viewed as representing the time-varying magnetic field impressed on the armature or stator. The voltage represents not a "signal" but a continuous voltage variable output directly. The distinction is not usually made, but the term "signal" is used more often when referring to sensor outputs.

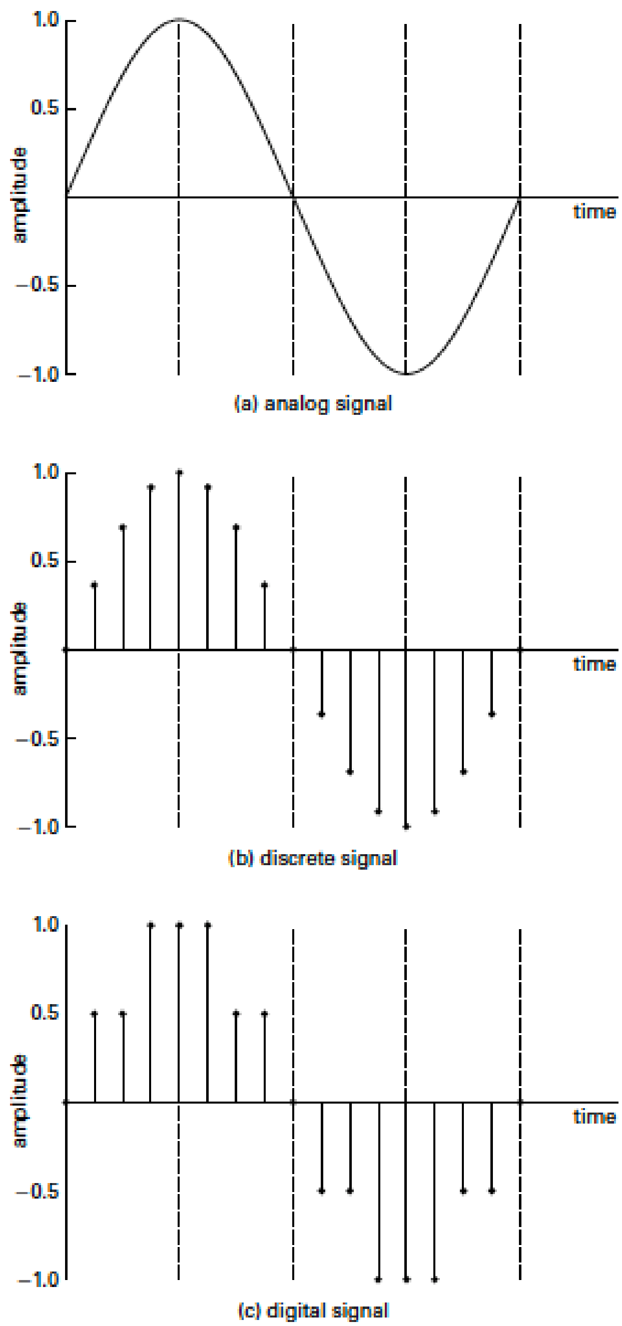


Figure 12: Representations of a Single AC Signal

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Appendix A: Equivalent Units Of Derived And Common SI Units

Symbol	Equivalent Units			
A	C/s	W/V	V/Ω	$J/(s \cdot V)$
C	$A \cdot s$	J/V	$(N \cdot m)/V$	$V \cdot F$
F	C/V	C^2/J	s/Ω	$(A \cdot s)/V$
F/m	$C/(V \cdot m)$	$C^2/(J \cdot m)$	$C^2/(N \cdot m^2)$	$s/(\Omega \cdot m)$
H	W/A	$(V \cdot s)/A$	$\Omega \cdot s$	$(T \cdot m^2)/A$
Hz	1/s	s^{-1}	cycles/s	$\text{radians}/(2\pi \cdot s)$
J	$N \cdot m$	$V \cdot C$	$W \cdot s$	$(kg \cdot m^2)/s^2$
m^2/s^2	J/kg	$(N \cdot m)/kg$	$(V \cdot C)/kg$	$(C \cdot m^2)/(A \cdot s^3)$
N	J/m	$(V \cdot C)/m$	$(W \cdot C)/(A \cdot m)$	$(kg \cdot m)/s^2$
N/A^2	$Wb/(N \cdot m^2)$	$(V \cdot s)/(N \cdot m^2)$	T/N	$1/(A \cdot m)$
Pa	N/m^2	J/m^3	$(W \cdot s)/m^3$	$kg/(m \cdot s^2)$
Ω	V/A	W/A^2	V^2/W	$(kg \cdot m^2)/(A^2 \cdot s^3)$
S	A/V	1/Ω	A^2/W	$(A^2 \cdot s^3)/(kg \cdot m^2)$
T	Wb/m^2	$N/(A \cdot m)$	$(N \cdot s)/(C \cdot m)$	$kg/(A \cdot s^2)$
V	J/C	W/A	C/F	$(kg \cdot m^2)/(A \cdot s^3)$
V/m	N/C	$W/(A \cdot m)$	$J/(A \cdot m \cdot s)$	$(kg \cdot m)/(A \cdot s^3)$
W	J/s	$V \cdot A$	V^2/Ω	$(kg \cdot m^2)/s^3$
Wb	$V \cdot s$	$H \cdot A$	T/m^2	$(kg \cdot m^2)/(A \cdot s^2)$

Appendix B: Physical Constants[1]

Quantity	Symbol	US Customary	SI Units
Charge			
electron	e		-1.6022×10^{-19} C
proton	p		$+1.6022\times10^{-19}$ C
Density			
air [STP][32°F, (0°C)]		0.0805 lbm/ft³	1.29 kg/m³
air [70°F, (20°C), 1 atm]		0.0749 lbm/ft³	1.20 kg/m³
sea water		64 lbm/ft³	1025 kg/m³
water [mean]		62.4 lbm/ft³	1000 kg/m³
Distance			
Earth radius[2]	⊕	2.09×10^{7} ft	6.370×10^{6} m
Earth-Moon separation[2]	⊕☾	1.26×10^{9} ft	3.84×10^{8} m
Earth-Sun separtion[2]	⊕⊙	4.89×10^{11} ft	1.49×10^{11} m
Moon radius[2]	☾	5.71×10^{6} ft	1.74×10^{6} m
Sun radius[2]	⊙	2.28×10^{9} ft	6.96×10^{8} m
first Bohr radius	a_0	1.736×10^{-10} ft	5.292×10^{-11} m
Gravitational Acceleration			
Earth [mean]	g	32.174 (32.2) ft/sec²	9.8067 (9.81) m/s²
Mass			
atomic mass unit	$\propto$ or $m_{\propto}$ $\frac{1}{12}m\left({}^{12}C\right)$	3.66×10^{-27} lbm	1.6606×10^{27} kg or 10^{-3} kg mol^{-1} / N_A
Earth[2]	⊕	4.11×10^{23} slugs	6.00×10^{24} kg
Earth [customary U.S.][2]	⊕	1.32×10^{25} lbm	-
Moon[2]	☾	1.623×10^{23} lbm	7.36×10^{22} kg
Sun[2]	⊙	4.387×10^{30} lbm	1.99×10^{30} kg
electron rest mass	m_e	2.008×10^{-30} lbm	9.109×10^{-31} kg
neutron rest mass	m_n	3.693×10^{-27} lbm	1.675×10^{-27} kg
proton rest mass	m_p	3.688×10^{-27} lbm	1.672×10^{-27} kg
Pressure			
atmospheric		14.696 (14.7) lbf/in²	1.0133×10^{5} Pa
Temperature			
standard		32°F (492°R)	0°C (273 K)
absolute zero		–459.67°F (0°R)	–273.16°C (0 K)
Velocity[3]			
Earth escape		3.67×10^{4} ft/sec	1.12×10^{4} m/s
light (vacuum)	c, c_0	9.84×10^{8} ft/sec	2.9979 (3.00) $\times10^{8}$ m/s
sound [air, STP]	a	1090 ft/sec	331 m/s

Table Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/.
2. Symbols shown for the solar system are those used by NASA. See https://science.nasa.gov/resource/solar-system-symbols/.
3. Velocity technically is a vector. It has direction.

Appendix C: Fundamental Constants

Quantity	Symbols	US Customary	SI Units
Avogadro's number	N_A, L		6.022×10^{23} mol^{-1}
Bohr magneton	∝B		9.2732×10^{-24} J/T
Boltzmann constant	κ	5.65×10^{-24} ft-lbf/°R	1.3805×10^{-23} J/T
electron volt: $\left(\frac{e}{C}\right)$ J	eV		1.602×10^{-19} J
Faraday constant, $N_A e$	F		96485 C/mol
fine structure constant, inverse α^{-1}	α α^{-1}		7.297×10^{-3} (≈1/137) 137.035
gravitational constant	g_c	32.174 lbm-ft/lbf-sec^2	
Newtonian gravitational constant	G	3.44×10^{-8} ft^4 / lbf-sec^4	6.672×10^{-11} $N\cdot m^2/kg^2$
nuclear magneton	∝N		5.050×10^{-27} J/T
permeability of a vacuum	∝0		1.2566×10^{-6} N/A^2 (H/m)
permittivity of a vacuum, electric constant $1/\mu_0 c^2$	ε0		8.854×10^{-12} $C^2/N\cdot m^2$ (F/m)
Planck's constant	h		6.6256×10^{-34} $J\cdot s$
Planck's constant: h/2π	□		1.0546×10^{-34} $J\cdot s$
Rydberg constant	R_∞		1.097×10^{7} m^{-1}
specific gas constant, air	R	53.3 ft-lbf/lbm-°R	287 $J/kg\cdot K$
Stefan-Boltzmann constant		1.71×10^{-9} BTU/ft^2-hr-°R^4	5.670×10^{-8} $W/m^2\cdot K^4$
triple point, water		32.02°F, 0.0888 psia	0.01109°C, 0.6123 kPa
universal gas constant	R*	1545 ft-lbf/lbmol-°R 1.986 BTU/lbmol-°R	8314 $J/kmol\cdot K$

Table Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/. The unit in Volume of "lbmol" is an actual unit, not a misspelling.

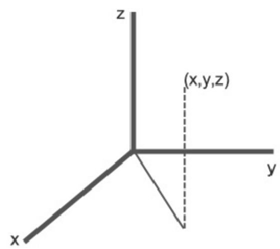
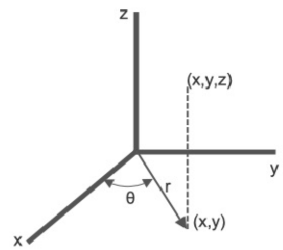
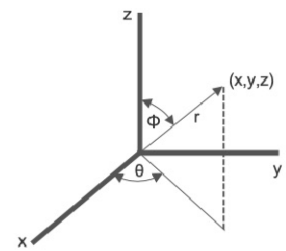
Appendix D: Mathematical Constants

Quantity	Symbol	Value
Archimedes' constant (pi)	π	3.1415926536
base of natural logs	e	2.7182818285
Euler's constant	C or τ	0.5772156649

Appendix E: The Greek Alphabet

Α	α	alpha	Ν	υ	nu
Β	β	beta	Ξ	ξ	xi
Γ	γ	gamma	Ο	o	omicron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	ϕ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

Appendix F: Coordinate Systems & Related Operations

Mathematical Operations	Rectangular Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Rectangular Coordinants	 $x = x$ $y = y$ $z = z$	 $x = r\cos\theta$ $y = r\sin\theta$ $z = z$	 $x = r\sin\phi\cos\theta$ $y = r\sin\phi\sin\theta$ $z = r\cos\phi$
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\theta + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\phi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \theta}\theta$
Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial (A_\phi \sin\phi)}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial A_\theta}{\partial \theta}$
Curl	$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \theta & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & A_\theta & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\mathbf{r} & \frac{1}{r^2\sin\theta}\phi & \frac{1}{r}\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ A_r & rA_\phi & rA\theta A_\phi \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r}\frac{\partial r}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial^2 f}{\partial \theta^2}\right)$